Contributions à la théorie des espaces de fonctions : singularités et relèvements

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Thèse de doctorat
sous la direction de **Petru Mironescu**

CONTRIBUTIONS À LA THÉORIE DES ESPACES DE FONCTIONS : SINGULARITÉS ET RELÈVEMENTS

Ioana A. Molnar

TAKE MY HOME, TAKE MY LAND
TAKE ME WHERE I CANNOT STAND
I DON'T CARE, I'M STILL FREE
YOU CAN'T TAKE THE SKY FROM ME
TAKE ME OUT TO THE BLACK
TELL THEM I AIN'T COMIN' BACK
BURN THE LAND AND BOIL THE SEA
YOU CAN'T TAKE THE SKY FROM ME
THERE'S NO OTHER PLACE I CAN BE
SINCE I FOUND SERENITY
⇾ Firefly : « The Ballad of Serenity », Joss Whedon.

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INTRODUCTION

Dans cette thèse nous nous sommes intéressés à deux aspects concernant les espaces de Sobolev des fonctions à valeurs dans la sphère unité : singularités et relèvements. Le manuscrit est divisé en deux chapitres qui correspondent chacun à l'un de ces deux sujets. Ainsi, le premier chapitre porte sur le problème de minimisation d'une énergie de Dirichlet à poids continu pour des fonctions $u \in \mathrm{W}^{1,n}$ à valeurs dans la sphère $\mathbb{S}^n$, aux singularités prescrites. Dans le deuxième chapitre nous étudions le problème des estimations optimales de relèvements des fonctions $u \in \mathrm{W}^{s,p}$ à valeurs dans le cercle $\mathbb{S}^1$. Chaque chapitre commence avec une introduction en français dans laquelle nous présentons le sujet, des résultats déjà connus dans le domaine et des généralisations et des résultats nouvelles que nous avons obtenus. Ils continuent avec une partie en anglais qui a fait le sujet d'une publication, respectivement.

Le premier chapitre, intitulé « *Le problème des singularités prescrites* », traite la relation entre les singularités des fonctions et l'infimum de leur énergies. Les fonctions concernées sont des applications u définies sur un domaine ouvert et régulier Ω de $\mathbb{R}^{m+n}$ qui prennent les valeurs dans la sphère unité $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ et qui appartiennent à l'espace de Sobolev $\mathrm{W}^{1,n}$. L'ensemble des singularités de ces fonctions, qui empêche qu'elles soient approchables par des fonctions lisses, est prescrit par le bord d'une variété m-dimensionnelle d'aire finie dans Ω ou, plus généralement, par le bord Γ d'un courant rectifiable de masse finie et de la même dimension. Fixé un tel Γ, la correspondance avec l'ensemble singulier de u se fait à l'aide du Jacobien généralisé, ou distributionnel, $\star \mathrm{J}u$ de u. Plus précisément, nous imposons la condition

$$\star \mathrm{J}u = \frac{\sigma_{n+1}}{n+1}\Gamma.$$

L'énergie utilisée dans ce problème est de la forme

$$E_{\Gamma,a}(u) \coloneqq \int_{\Omega\smallsetminus\Gamma} a(x)\,|\mathrm{D}u(x)|^n\,\mathrm{d}x.$$

Il s'agit donc d'une énergie de type Dirichlet avec un poids $a(\cdot)$ positive et *a priori* mesurable, mais qu'on va considérer continue. La question que nous nous sommes posée est de minimiser $E_{\Gamma,a}$.

Ce problème à été étudié d'abord par Brezis, Coron et Lieb (1986) dans le cas particulier où $n = 2$, $m = 1$, $a(\cdot) \equiv 1$. Alors les singularités sont donnés par un ensemble fini des points a_i dans Ω et on fixe le degré topologique de u autour de chacun de ces points (comme application entre des sphères de $\mathbb{R}^3$) ou, équivalent, on associe aux a_i des multiplicités entières. Le minimum de l'énergie dans ce cas est donné par

$$8\pi\ \min\{\ell(C) \mid C = \text{connexion qui joint les singularités de } u\},$$

où les *connexions* sont définies d'une manière rigoureuse en tenant compte de la multiplicité des singularités ponctuelles a_i.

Puis, Almgren, Browder et Lieb (1988), Alberti, Baldo et Orlandi (2000) et Millot (2005) ont continué l'étude de ce problème pour des dimensions n et m quelconques et pour une énergie avec poids. Ainsi le résultat initial a été généralisé successivement, aussi que le cadre fonctionnel où le problème et placé et les méthodes pour estimer l'énergie. Enfin, en adaptant les méthodes utilisées dans les travaux mentionnés, nous montrons que

$$\inf E_{\Gamma,a}(u) = n^{n/2}\sigma_{n+1} \inf \mathbb{M}(M \llcorner a)$$

où l'infimum du membre droite est pris pour les courants rectifiables M de dimension $m+n$ de bord égale à Γ. Ce résultat représente, naturellement, une généralisation des résultats connus précédemment.

Le deuxième chapitre, appelé « *Relèvements des applications à valeurs dans le cercle* » est centré sur le problème de trouver les meilleures estimations des relèvements des applications unimodulaires, en termes des semi-normes et dans le cadre des espaces de Sobolev fractionnaires $\mathrm{W}^{s,p}$. Plus précisément, nous considérons cette fois les fonctions u définies sur un domaine ouvert et régulier de $\mathbb{R}^n$ à valeurs dans $\mathbb{C}$ et de module 1.

Le point de départ représente le travail de Brezis, Bourgain et Mironescu (2000, 2002) qui ont investigué d'abord les espaces $\mathrm{W}^{s,p}$ pour lesquelles chaque $u \in \mathrm{W}^{s,p}(\Omega;\mathbb{S}^1)$ admet une fonction $\varphi \in \mathrm{W}^{s,p}(\Omega;\mathbb{R})$ telle que $u = \exp(\mathrm{i}\varphi)$: appelé relèvement, argument ou phase de u. Les auteurs ont complètement répondu à cette question selon les valeurs de s, p et n. Il se distinguent trois régimes différentes de l'existence du relèvement : $sp < 1$ et $s < 1$ (preuve constructive), $sp \geqslant n$ et $s < 1$, et finalement $sp \geqslant 2$ avec $s > 1$ (unicité du relèvement). Il s'impose alors la question de l'estimation de la semi-norme $|\varphi|_{\mathrm{W}^{s,p}}$ en fonction de $|u|_{\mathrm{W}^{s,p}}$, voir une estimation optimale (en général à une constante indépendante de s près).

Nous présentons ici l'un de plusieurs résultats que nous avons obtenus. Il s'agit du cas le plus délicat : $sp < 1$, $s < 1$. L'estimation de la forme

$$|\varphi|_{\mathrm{W}^{s,p}} \leqslant C(n,p)\frac{1}{s(1-sp)}|u|_{\mathrm{W}^{s,p}},$$

est valable pour tout $p \geqslant 1$ et optimale pour $p > 1$. Ce résultat représente une amélioration de l'estimation fournit par la construction dyadique proposée dans la preuve de l'existence. En revanche, il est obtenu à partir de cette construction à l'aide d'une méthode des moyennes inspirée par le travail de Garnett et Jones (1982). Cette estimation était déjà connue dans le cas $p = 2$ par Brezis, Bourgain et Mironescu mais l'utilisation de l'analyse de Fourier ne permettait pas une généralisation directe au cas p quelconque. Par des techniques nouvelles, nous la démontrons aussi pour $p \neq 2$. L'optimalité avait été obtenue aussi dans le cas $p = 2$ par une preuve élaborée qui utilise, parmi d'autres outils, le comportement des constantes optimales dans des plongements de Sobolev. Nous simplifions cette preuve à l'aide d'une inégalité de réarrangement de Garsia et Rodemich (1974) qui est en plus valable pour tout $p > 1$. Finalement, dans le cas $p = 1$, l'estimation optimale est donnée par

$$|\varphi|_{\mathrm{W}^{s,p}} \leqslant C(n,p)2|u|_{\mathrm{W}^{s,p}},$$

qu'on obtient en utilisant l'approche de Dàvila et Ignat (2003) pour les relèvements BV. En plus, nous avons donné des estimations optimales aussi dans les deux cas restantes, des fois en généralisant les cas particuliers connus, des fois en utilisant des techniques nouvelles.

TABLE DES MATIÈRES

CHAPITRE 1 LE PROBLÈME DES SINGULARITÉS PRESCRITES

1.1. Présentation du problème

Soient Ω un domaine ouvert et régulier strictement inclus dans $\mathbb{R}^3$, a un point fixé dans Ω et d un nombre entier. Considérons les applications $u\colon \Omega \to \mathbb{S}^2$ qui sont de classe C^1 sur Ω sauf au point a. $\mathbb{S}^2$ désigne la sphère unité de $\mathbb{R}^3$.

La singularité topologique unique d'une telle fonction u est prescrite par sa position dans Ω et par son degré. Ici, le degré au point a de u représente le degré topologique de u vu comme une application continue de $\mathbb{S}^2$ en $\mathbb{S}^2$. Plus précisément, nous restreignons u à une petite sphère $\partial B^3(a,r) \subset \Omega$, ensuite nous composons $u\,|_{\partial B^3(a,r)}$ avec une homothétie affine en vue d'obtenir une application $\widetilde{u} \in \mathrm{C}(\mathbb{S}^2;\mathbb{S}^2)$, et nous posons $\deg(u,a) \coloneqq \deg \widetilde{u}$. Cette quantité ne dépend pas du choix du rayon r. Par ailleurs (voir [**15**]), si $\widetilde{u}$ est en plus de classe C^1, le degré de $\widetilde{u}$ peut être donné par la formule explicite

$$\deg \widetilde{u} = \frac{1}{\mathcal{H}^2(\mathbb{S}^2)} \int_{\mathbb{S}^2} \det \mathrm{J}\widetilde{u}(x)\, \mathrm{d}\mathcal{H}^2(x),$$

où $\mathcal{H}^2$ désigne la mesure de Hausdorff 2-dimensionnelle et $\mathrm{J}u$ est le determinant de la 2×2-matrice Jacobienne de $\widetilde{u}$.

La solution du problème de minimisation de l'énergie Dirichlet de u est

$$\inf \left\{ \int_\Omega |\mathrm{D}u|^2(x)\, \mathrm{d}x \,\middle|\, \deg(u,a) = d \right\} = 8\pi\, |d|\, \operatorname{dist}(a, \partial\Omega), \tag{1.1}$$

où $|\mathrm{D}u|$ est la norme Euclidienne de ∇u.

Il faut préciser que le problème est trivial si $\Omega = \mathbb{R}^3$. Dans ce cas-ci, pour que $\displaystyle\int_{\mathbb{R}^3} |\mathrm{D}u|^2$ soit fini, le degré de u doit être forcément nul, ce qui permet ainsi que u soit continue sur $\mathbb{R}^3$ est donc l'infimum dans (1.1) est nul.

Le résultat (1.1) a été démontré par Brezis, Coron et Lieb dans [**16**]. Les auteurs ont aussi étudié et démontré plusieurs généralisations de ce problème que nous présentons dans la suite.

Supposons maintenant que nous avons deux singularités $a_1, a_2 \in \mathbb{R}^3$ aussi que leur degrés (entiers) associés $d_1 \leqslant d_2$ qui vérifient

$$d_1 + d_2 = 0. \tag{1.2}$$

Les fonctions admissibles sont alors $u \in \mathrm{C}(\mathbb{R}^3 \setminus \{a_1, a_2\}; \mathbb{S}^2)$ et l'énergie minimale est

$$\inf\left\{\int_{\mathbb{R}^3} |\mathrm{D}u|^2(x)\,\mathrm{d}x \;\middle|\; \deg(u,a_1) = -\deg(u,a_2) = d\right\} = 8\pi\,|d|\,|a_1 - a_2|. \tag{1.3}$$

La condition (1.2) est nécessaire, comme avant, pour que $\int_{\mathbb{R}^3} |\mathrm{D}u|^2 < \infty$. En effet, d'une part le degré de u restreint à une sphère de rayon assez grand doit être nul, et d'un autre part il représente la somme des degrés de u aux points a_1 et a_2.

Par ailleurs, l'infimum de (1.3) n'est pas atteint : si u_n est une suite minimisante, alors $|\mathrm{D}u_n|^2$ converge *-faiblement (au sens des mesures) vers la mesure de Hausdorff sur le segment $[a_1, a_2]$ ([**16**, section VI]).

Si nous considérons le problème sur $\Omega \subsetneq \mathbb{R}^3$, alors (1.2) n'est plus nécessaire, et nous avons le résultat

$$\begin{aligned} &\inf\left\{\int_{\Omega} |\mathrm{D}u|^2(x)\,\mathrm{d}x \;\middle|\; \deg(u,a_i) = d_i,\ i = 1,2\right\} \\ &= 8\pi \begin{cases} d_m\,|a_1 - a_2| + |d_1 + d_2|\ \operatorname{dist}(a_M, \partial\Omega) & \text{si} \quad \operatorname{sgn} d_1 = -\operatorname{sgn} d_2 \\ |d_1|\ \operatorname{dist}(a_1, \partial\Omega) + |d_2|\ \operatorname{dist}(a_2, \partial\Omega) & \phantom{\text{si}} \quad \operatorname{sgn} d_1 = \operatorname{sgn} d_2 \end{cases}, \end{aligned} \tag{1.4}$$

où $d_m \coloneqq \min\{|d_1|, |d_2|\}$, $d_M \coloneqq \max\{|d_1|, |d_2|\}$ et a_M est la singularité a_i telle que $|d_i| = d_M$. On verra qu'en réalité les formules (1.3) et (1.4) ont une expression commune plus simple dans le cas général.

Pour le cas de n singularités $a_1, \ldots, a_n \in \mathbb{R}^3$, supposons d'abord que les degrés $d_1, \ldots, d_n \in \mathbb{Z}$ satisfont la condition

$$\sum_{i=1}^{n} d_i = 0. \tag{1.5}$$

L'étude du problème de minimisation associé conduit à la notion de *longueur d'une connexion minimale*, définie en [**16**] de la manière suivante.

Nous commençons par nommer les a_i des points positifs ou des points négatifs selon le signe de leur d_i ; les points de degré nul ne jouent (à nouveau) aucun rôle. Puis nous formons des couples « (point négatif, point positif) » de sorte que chaque point a_i appartienne à $|d_i|$ paires ; ici nous voyons bien le rôle de (1.5). La collection résultante est appelée connexion admissible pour la configuration donnée de points et de degrés. Sur l'ensemble de toutes les connexions $C = \{(a_i, a_j)\}_{i,j}$ admissibles, la longueur L est définie ensuite comme la somme des longueurs des segments associés aux éléments de C :

$$L(C) = \sum_{i,j} |a_i - a_j|. \tag{1.6}$$

Finalement, nous appelons *connexion minimale* une connexion qui minimise $L(\cdot)$. Donc une connexion minimale est l'une qui joint de la façon la moins coûteuse les points $a_1, \ldots, a_n$ en tenant compte de leur multiplicité (fournie par leur degrés topologiques associés).

Le résultat du problème de minimisation de l'énergie pour les fonctions admissibles $u \in \mathrm{C}(\mathbb{R}^3 \setminus \{a_i\}_{i=1}^n; \mathbb{S}^2)$ est

$$\inf\left\{\int |\mathrm{D}u|^2(x)\,\mathrm{d}x \;\middle|\; \deg(u,a_i) = d_i\right\} = 8\pi \min_{C} L(C). \tag{1.7}$$

Dans le cas d'un domaine $\Omega \subsetneq \mathbb{R}^3$, nous n'avons plus besoin de supposer (1.5). Le raison pour cela est que le bord de Ω tient place d'une singularité de degré

$d \coloneqq -\sum_{i=1}^{n} d_i$. Soit une connexion admissible pour une telle configuration. Aux paires qui contient la « singularité » $\partial\Omega$, nous donnons le sens suivant : $(a_i, \partial\Omega) = (a_i, P_{\partial\Omega} a_i)$ avec $P_{\partial\Omega} a$ représentant la projection du point $a \in \Omega$ sur $\partial\Omega$. Avec cette interprétation, la formule (1.6) garde son sens ; en plus, cela explique la formule (1.4). Pour les fonctions admissibles $u \in \mathrm{C}(\Omega \setminus \{a_i\}_{i=1}^{n}; \mathbb{S}^2)$ l'énergie minimale est encore donnée par la formule (1.7).

La généralisation à une énergie à poids a été étudiée par Millot en [**39**]. Le poids est donné par une fonction $a : \Omega \to \mathbb{R}$ qui est mesurable, bornée et minorée par une constante strictement positive. Millot a obtenu, dans ce cadre, l'analogue de (1.6). Sa formule fait intervenir une connexion minimale *ad hoc*, prenant en compte la présence de $a(\cdot)$. Dans la définition (1.6), la distance $|a_i - a_j|$ est remplacé par l'infimum des longueurs pondérées des lignes polygonales reliant a_i à a_j. La longueur pondérée d'une arête $[p, q]$ est définie par l'intégrale moyenne de la fonction $a(\cdot)$ sur l'ensemble $\Xi(p, q; \varepsilon)$ obtenue à partir du segment $[p, q]$ auquel nous rajoutons une épaisseur $\varepsilon \to 0$:

$$\ell(p, q) \coloneqq \liminf_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{\Xi(p,q;\varepsilon)} a(x)\, \mathrm{d}x.$$

Retournons au problème sans poids, nous considérons des extensions en dimensions supérieures. Dans cette direction, la première extension naturelle de (1.7) à été déjà étudiée dans [**16**]. Elle fait intervenir des fonctions $u\colon \mathbb{R}^p \to \mathbb{S}^{p-1}$ et l'énergie

$$E(u) \coloneqq \int_{\Omega} |\mathrm{D}u|^{p-1}(x)\, \mathrm{d}x.$$

L'infimum est pris à nouveau sur l'ensemble des fonctions u continues sauf aux points $a_1, \ldots, a_n \in \Omega$ prescrits par leurs degrés $d_1, \ldots, d_n$ de somme nulle. Dans ce cadre, nous avons le résultat

$$\inf E(u) = \mathcal{H}^p(\mathbb{S}^p)(p-1)^{(p-1)/2} \min_{C} L(C).$$

Un autre type d'extension de (1.7) est de changer la dimension de l'espace où vivent les fonctions u. Par example, soit Ω un domaine de $\mathbb{R}^3$ et Γ une courbe de Jordan rectifiable et orientée dans Ω. Nous considérons les fonctions u à valeurs dans $\mathbb{S}^1$, continues en dehors de Γ. Ici, la notion de degré de u le long de Γ se définit de manière suivante : soit $x \in \Gamma$ et soit $\mathcal{C}$ un petit disque orienté qui intersecte transversalement Γ au point x et dont l'orientation est consistante avec celle de Ω (c.-à-d. si $\mathrm{T}_x\Gamma \eqqcolon \mathrm{Vect}\{\tau_1, \tau_2\}$ et $\mathrm{T}_x\mathcal{C} \eqqcolon \mathrm{Vect}\{\mu\}$, alors (τ_1, τ_2, μ) est un repère direct de $\mathbb{R}^3$). Nous prenons ensuite $d = \deg(u, \Gamma) \coloneqq \deg(u\,|_{\mathcal{C}})$. Cette quantité ne dépend ni du rayon de $\mathcal{C}$ ni du choix de x. En plus, dans cette situation, la valeur de $d \in \mathbb{Z}$ n'est soumise à aucune contrainte.

L'énergie minimale associée à telles applications u est alors

$$\inf \int_{\Omega} |\mathrm{D}u|(x)\, \mathrm{d}x = 2\pi |d| \inf \left\{ \mathcal{H}^2(\mathcal{S}) \,\middle|\, \mathcal{S} \text{ surface de bord } \partial\mathcal{S} = \Gamma \right\} \tag{1.8}$$

([**16**] pour Γ courbe plane).

Le problème (1.8) se pose aussi en dimensions et codimensions arbitraires. Pour Ω un domaine de $\mathbb{R}^{m+n}$, M une variété $(m-1)$-dimensionnelle sans bord dans Ω avec degré associé $d \in \mathbb{Z}$, et pour des fonctions $u : \Omega \to \mathbb{S}^n$ qui ont M pour ensemble singulier, quelle est l'énergie minimale

$$E_M(u) \coloneqq \int_\Omega |\mathrm{D}u|^n(x)\,\mathrm{d}x? \tag{1.9}$$

Quand M est polygonal, Almgren, Browder and Lieb ([**4**]) donnent la réponse et une esquisse de preuve à l'aide des courants entiers (qui sont des généralisation des surfaces) :

$$\inf E_M(u) = n^{n/2}\mathcal{H}^n(\mathbb{S}^n)\inf\{\mathbb{M}(T) \mid T \text{ courant entier avec } \partial T = M\}, \tag{1.10}$$

où $\mathbb{M}(T)$ représente la masse d'un courant T.

Plus tard, le problème lui-même est posé dans le contexte des courants rectifiables par Alberti, Baldo et Orlandi ([**3**]). Leur but était d'étudier l'image du Jacobien distributionnel (ou généralisé) $\mathrm{J}u$ pour les fonctions u appartenant à l'espace $\mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$. Les auteurs ont montré une identification entre $\mathrm{J}u$ et le bord d'un courant rectifiable T de codimension n dans Ω. D'une part, pour chaque u, il existe un T tel que

$$\mathrm{J}u = \partial T\,\frac{1}{n+1}\mathcal{H}^n(\mathbb{S}^n) \tag{1.11}$$

et qui vérifie

$$c(n,m)\mathbb{M}(T) \leqslant \int_\Omega |\mathrm{D}u|^n(x)\,\mathrm{d}x. \tag{1.12}$$

D'autre part, étant donné T, il existe une application u qui vérifie (1.11) et

$$\int_\Omega |\mathrm{D}u|^n(x)\,\mathrm{d}x \leqslant C(n,m)\mathbb{M}(T). \tag{1.13}$$

En combinant les résultats présentés ci-dessus, les idées et méthodes utilisées dans leur preuves, nous avons amélioré (1.12) et (1.13) en montrant que la formule (1.10) reste vraie dans le contexte général des courants rectifiables et au cas d'une énergie à poids continu.

1.2. Le Jacobien distributionnel $\mathrm{J}u$

Le Jacobien distributionnel à été introduit, sous la forme que nous utilisons ici, dans [**33**]. Pour une fonction $u\colon \Omega \to \mathbb{R}^{n+1}$, avec $\Omega \subset \mathbb{R}^p$, le Jacobien distributionnel $\mathrm{J}u$ est la $(n+1)$-forme différentielle sur Ω définie (au sens distributionnel) par :

$$\mathrm{J}u \coloneqq \frac{1}{n+1}\,\mathrm{d}(u^\sharp\omega), \tag{1.14}$$

dès que cette formule est bien définie. Ici, $u^\sharp\omega$ est le tiré en arrière par $\mathrm{d}u$ de la forme volume ω_0 sur la sphère $\mathbb{S}^n$. Plus précisément, pour $y \in \mathbb{S}^n$,

$$\omega(y) \coloneqq \sum_{i=1}^{n+1}(-1)^{i-1}y_i\,\widehat{\mathrm{d}y_i} \coloneqq \sum_{i=1}^{n+1}(-1)^{i-1}y_i\,\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}y_{i_1}\wedge\mathrm{d}y_{i+1}\wedge\cdots\wedge\mathrm{d}y_{n+1},$$

donc (1.14) peut se récrire sous la forme

$$\mathrm{J}u = \frac{1}{n+1}\,\mathrm{d}\sum_{\alpha\in I(n,p)}\det(\partial_{\alpha_1}u,\ldots,\partial_{\alpha_n}u,u)\,\mathrm{d}x_{\alpha_1}\wedge\cdots\wedge\mathrm{d}x_{\alpha_n} \tag{1.15}$$

$\big($où $I(n,p)$ est l'ensemble usuel de multi-indices ordonnés $\alpha = (\alpha_1,\ldots,\alpha_n) \in [\![1,p]\!]^n\big)$.

La formule (1.15) est bien définie si $u \in \mathrm{W}^{1,n} \cap \mathrm{L}^\infty(\Omega; \mathbb{R}^{n+1})$. Si, en plus, $u \in \mathrm{W}^{1,n+1}$, alors nous pouvons différentier chaque terme de la somme du membre droite de (1.15) et donc

$$(1.16) \quad \mathrm{J}u = \sum_{\alpha \in I(n+1,p)} (\partial_{\alpha_1} u, \dots, \partial_{\alpha_{n+1}} u)\, \mathrm{d}x_{\alpha_1} \wedge \cdots \wedge \mathrm{d}x_{\alpha_{n+1}};$$

pour un tel u défini sur $\Omega \subset \mathbb{R}^{n+1}$, cette formule revient au Jacobien classique $\mathrm{J}u = \det(\mathrm{D}u)\,\mathrm{d}x$.

D'autre part, si u appartient à $\mathrm{W}^{1,n+1}$ et prend ses valeurs dans $\mathbb{S}^n$, alors $\mathrm{J}u = 0$ (les determinants dans (1.16) étant tous nuls). Dans le cas général, quand $u \in \mathrm{W}^{1,n}(\Omega; \mathbb{R}^{n+1})$, le Jacobien distributionnel $\mathrm{J}u$ décrit, en un sens, les singularités topologiques de u. Les résultats suivants décrivent cette propriété.

Le premier résultat dans cette direction appartient à Bethuel. Dans [**7**], Bethuel montre que u n'a pas de singularités topologiques si (et seulement si) le jacobien de u est nul.

Théorème ([**7**]). — *Soit $\Omega \subset \mathbb{R}^{n+1}$ et $u \in \mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$. Alors u peut être approchée dans la norme $\mathrm{W}^{1,n}$ par des fonctions régulières $u_j : \Omega \to \mathbb{S}^n$ si et seulement si $\mathrm{J}u = 0$.*

Souvent, il est plus convenable d'utiliser le courant $\star\mathrm{J}u$ à la place de la forme différentielle $\mathrm{J}u$. L'opérateur $\star$ représente l'opérateur de Hodge et transforme une $(n+1)$-forme en $(p-n-1)$-courant. Le théorème, enoncé et utilisé dans [**3**], donne la formule de $\star\mathrm{J}u$ dans le cas où u est continue hors d'une sous-variété.

Théorème. — *Soient $\Omega \subset \mathbb{R}^p$ et $u \in \mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$ continue en-dehors d'une sous-variété $(p-n-1)$-dimensionnelle S. Alors $\star\mathrm{J}u = \dfrac{\mathcal{H}^n(\mathbb{S}^n)}{n+1} \deg(u, S)\, S$.*

Ici, nous notons par S l'intégration sur S. La preuve de ce résultat n'apparaît pas dans la littérature lorsque p, n et S sont arbitraires. Toutefois, le résultat a été démontré dans des cas particuliers : pour $p = n+1$ dans [**16**], pour $n = 2$ et S un disque $(p-3)$-dimensionnel par Pakzad, [**49**], pour $p = 3$ et $n = 1$ par Jerrard et Soner ([**33**]).

La variante suivante du théorème précédent a, en revanche, été établie pour tous n, p et S.

Théorème. — *(voir [**3**, Theorem 3.8]) Soient $\Omega \subset \mathbb{R}^p$ et $u \in \mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$. Soit N_y le courant rectifiable associé à l'ensemble de niveau $u^{-1}(y)$ dont on enlève les points où u n'est pas approximativement différentiable.*

Alors $\star\mathrm{J}u = (-1)^{p-n} \dfrac{\mathcal{H}^n(\mathbb{S}^n)}{n+1} \partial N_y$, pour $\mathcal{H}^n$-presque tous $y \in \mathbb{S}^n$.

Un autre résultat qui peut se révéler utile dans ce contexte est :

Théorème ($\star$). — *Soit $\Omega \subset \mathbb{R}^p$ un domaine borné, régulier et connexe, et soit $u \in \mathrm{W}^{1,n}(\Omega; \mathbb{S}^n)$. Alors u peut être approchée dans la norme $\mathrm{W}^{1,n}$ par des fonctions $u_j : \Omega \to \mathbb{S}^n$ régulières sauf sur $\Gamma_j \coloneqq \partial u_j^{-1}(y)$, pour $\mathcal{H}^n$-presque tous $y \in \mathbb{S}^n$.*

Un résultat similaire peut être trouvé dans [**8**, Théorème 4]. Une preuve de ce résultat qui est inspirée par celle du résultat d'approximation de Bousquet ([**14**, section 5]), est donnée dans la section 1.5 à la fin du chapitre.

1.3. Aperçu du résultat

Le résultat que nous avons obtenu est le suivant (voir [**46**, Theorem T1]).

Soit $\Omega \subset \mathbb{R}^{m+n}$ un domaine borné et connexe. Soit $\Gamma = \partial M_0$ le bord d'un courant rectifiable $M_0 \subset \Omega$ de dimension $\dim M_0 = m$ et masse $\mathbb{M}(M_0)$ finie. Nous considérons la classe suivante de fonctions admissibles :

$$\mathcal{E}(\Gamma) \coloneqq \left\{ u \in \mathrm{W}^{1,n}(\Omega; \mathbb{S}^n) \;\middle|\; \star\, \mathrm{J}u = \frac{\mathcal{H}^n(\mathbb{S}^n)}{n+1}\, \Gamma \right\}.$$

A chaque fonction $u \in \mathcal{E}(\Gamma)$ nous associons l'énergie à poids suivante :

$$E_\Gamma(u,a) \coloneqq \int_\Omega a(x) |\mathrm{D}u(x)|^n \,\mathrm{d}x,$$

où le poids $a : \Omega \to \mathbb{R}_+$ est une fonction continue qui vérifie $\inf a > 0$.

Considérons aussi le sous-ensemble suivant des courants rectifiables

$$\mathcal{C}(\Gamma) \coloneqq \{ M \in \mathcal{R}(\mathbb{R}^{m+n}) \mid \partial M = \Gamma, \mathbb{M}(M) < \infty \}.$$

Alors, la formule de l'énergie minimale est donné par :

$$\inf_{u \in \mathcal{E}(\Gamma)} E_\Gamma(u,a) = n^{n/2} \mathcal{H}^n(\mathbb{S}^n) \inf_{M \in \mathcal{C}(\Gamma)} \mathbb{M}(M \llcorner a), \tag{1.17}$$

où $M \llcorner a$ signifie que la fonction de multiplicité du courant M est multipliée par la fonction mesurable $a(\cdot)$.

La preuve consiste à montrer l'égalité (1.17) par double inégalité, donc elle est divisée en deux parties. La borne supérieure de l'énergie est obtenue à l'aide d'une construction d'un « dipôle », idée introduite dans [**16**]. Si les singularités sont des points, alors un dipôle représente un presque-minimiseur de l'énergie qui est concentré autour des segments $[a,b]$ appartenant à une connexion minimale. La construction se généralise aisément dans le cas où l'ensemble singulier est une courbe plane. L'extension au cas du bord d'un courant rectifiable utilise un résultat d'approximation par des courants polyédraux et un raisonnement par induction, méthode présentée dans [**4**]. Nous introduisons un type de mesure de Hausdorff à poids $\mathcal{H}^n_a$ qui permet d'obtenir les mêmes résultats pour l'énergie avec un poids continu $a(\cdot)$. Cette mesure définie *ad hoc* constitue une compensation pour le fait qu'à cause de $a(\cdot)$, les fonctions vivent dans un cadre hétérogène. L'idée de modifier la mesure classique est similaire à celle vue dans [**39**] pour un poids mesurable. Il faut remarquer que, sous la forme présentée, notre preuve ne marche que dans le cas particulier où $a(\cdot)$ est continue. Par contre, la définition de $\mathcal{H}^n_a$ a un sens si $a(\cdot)$ est seulement mesurable, et donc elle pourrait représenter l'outil pour la preuve dans ce cas plus général.

Pour la borne inférieure, la preuve repose sur une formule de type co-aire présentée dans [**3**] et inspirée par [**4**].

1.4. Preuve du résultat

Cette section contient le travail qui fait l'objet d'une publication en novembre 2012 ([**46**]).

Prescribed singularities with weights

Abstract: We find the minimal weighted energy $\int_\Omega a(x)|\mathrm{D}u(x)|^n\,\mathrm{d}x$ *of maps* $u:\Omega\to\mathbb{S}^n$ *with prescribed singularities, where* $\Omega\subset\mathbb{R}^{m+n}$ *and* $a(\cdot)$ *is a continuous positive weight. Our result extends previous ones of Brezis, Coron and Lieb (1986), Alberti, Baldo and Orlandi (2003), and Millot (2005).*

1.4.1. Introduction. — The problem of determining the minimum energy of a map u with values in the unit sphere and with prescribed singularities was first investigated by Brezis, Coron, and Lieb in *Harmonic Maps with Defects* ([**16**]), and it can be seen as a starting point in the analysis of some problems with applications in physics, like the ones concerning liquid crystals. The *two-point problem* was the following: given two points A_1 and A_2 in Ω, and a positive integer $d\in\mathbb{Z}_+$,

$$\text{minimize } E_{A_i,d_i}(u) := \int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x,\quad \text{when } u\in \mathrm{C}(\Omega\setminus\{A_1,A_2\}\subset\mathbb{R}^3;\mathbb{S}^2),$$
$$\text{and } \deg(u,A_1) = -\deg(u,A_2) = d,$$

where $\deg(u,A_i)$ is the usual topological degree of the restriction of u to a small sphere $S\subset\Omega$ surrounding the point A_i (and is independent of the specific choice of S). The answer to the problem was given as

$$\inf_u E_{A_i,d_i}(u) = 8\pi d\,\mathrm{dist}(A_1,A_2),$$

with the infimum not being, in general, achieved.

In this paper, we will treat a minimal energy problem that arises from a sequence of generalizations of the one above. Several variations were already proposed in [**16**], and the following three are of interest to us. The first one was to consider more than two singularities (still a finite number of them), which lead to the concept of *minimal connection* between singularities with assigned degree. It was shown ([**16**, Theorem 1.1]) that the solution of the problem corresponding to the points $A_1,\dots,A_N$ and the degrees $d_1,\dots,d_N\in\mathbb{Z}$ that satisfy $\sum_i d_i = 0$, is

$$\inf_u E_{A_i,d_i}(u) = 8\pi L,\quad \text{where } L \text{ is the length of a minimal connection.} \tag{1.18}$$

The key step in proving the inequality " $\leqslant$ " is the *dipole construction*, that is, the construction of an almost minimizer concentrated around each line segment associated to a minimal connection.

A second generalization consisted in placing the problem in a higher dimension, taking $u\in\mathrm{C}(\Omega\setminus\{A_1,\dots,A_N\}\subset\mathbb{R}^p\ ;\ \mathbb{S}^{p-1})$, and it was proved that the least energy is given in this case by

$$\inf_u \int_{\mathbb{R}^p} |\mathrm{D}u(x)|^{p-1}\,\mathrm{d}x = \sigma_p(p-1)^{(p-1)/2}L, \tag{1.19}$$

where σ_p represents the $(p-1)$-dimensional measure of the sphere $\mathbb{S}^{p-1} \subset \mathbb{R}^p$. The third extension of the two-point problem, relevant here, was to consider a situation where the energy has the homogeneity of an area ([**16**, section VIII.C]), for example minimize

$$E_\Gamma(u) := \int_{\mathbb{R}^3} |\mathrm{D}u(x)|\,\mathrm{d}x, \quad \text{for } u \in \mathrm{C}(\Omega \smallsetminus \Gamma \subset \mathbb{R}^3\ ;\ \mathbb{S}^1) \ \text{ with } \deg(u,\Gamma) = d,$$

where Γ is a given rectifiable, oriented Jordan curve in $\mathbb{R}^3$, and $d \in \mathbb{Z}$ is fixed. In the special case where the curve is planar, it was proved that

$$\inf_u E_\Gamma(u) = 2\pi|d| \inf\{\mathrm{area}(\mathcal{S})\ ;\ \mathcal{S} \text{ surface in } \mathbb{R}^3,\ \partial\mathcal{S} = \Gamma\}. \tag{1.20}$$

Moreover, the authors raise the same question in arbitrary dimension and codimension, more specifically

$$\begin{aligned} &\text{minimize } E_{M_0}(u) := \int_\Omega |\mathrm{D}u(x)|^n\,\mathrm{d}x, \\ &\text{for } u \in \mathrm{C}(\Omega \smallsetminus M_0 \subset \mathbb{R}^{m+n}; \mathbb{S}^n), \ \text{ and } \deg(u, M_0) = d, \end{aligned} \tag{1.21}$$

where M_0 is an $(m-1)$-dimensional boundaryless manifold in $\mathbb{R}^{m+n}$, and $d \in \mathbb{Z}$. Here, $\deg(u, M_0)$ represents the degree of the restriction of u to a n-dimensional sphere S that *links* with M_0 (i.e., S is the boundary of a well-oriented $(n+1)$-dimensional disk that transversally intersects M in a single point), and does not depend on the choice of S. They suggest that the solution should have the same form as (1.20), but the formula was afterwards rectified by Almgren, Browder, and Lieb in the paper *Co-Area, Liquid Crystals, and Minimal Surfaces* ([**4**]).

In [**4**], the authors first provide a new proof of the inequality " $\geqslant$ " in (1.18), which uses the coarea formula and current slicing, and which can more easily be extended to higher dimensions. In the main theorem, they give the solution of the problem (1.21), put in the context of integral currents, and then give an outline of the proof. The result obtained was, roughly, that

$$\inf_u E_{M_0}(u) = n^{n/2}\sigma_{n+1} \inf\{\mathrm{mass}(T)\ ;\ T \text{ is an integral current with } \partial T = d\,M_0\}.$$

A similar setting for this problem is given in the article by Alberti, Baldo and Orlandi entitled *Functions with prescribed singularities* ([**3**]). Their main interest was to study the image of the distributional Jacobian $\mathrm{J}u$, which is known to describe in some sense the topological singularities of the map u. They have shown an identification between $\star\mathrm{J}u$, the current associated, via the Hodge-type operator $\star$, to the distributional Jacobian of a $\mathbb{S}^n$-valued map, and the boundary of a rectifiable current of codimension n. More precisely (see [**3**, Theorem 3.8 and Theorem 5.6]), they have proved, on one side, that given $u \in \mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$, there exists a rectifiable current T that satisfies the inequality

$$\sigma_{n+1} \times \mathrm{mass}(T) \leqslant \int_\Omega |\mathrm{D}u(x)|^n\,\mathrm{d}x \tag{1.22}$$

and the condition $\star\mathrm{J}u = \dfrac{\sigma_{n+1}}{n+1}\partial T$, and, conversely, that for a given current T, there exists a map u that verifies the above condition on its distributional Jacobian, and is such that

$$\int_\Omega |\mathrm{D}u(x)|^n\,\mathrm{d}x \leqslant c(m,n) \times \mathrm{mass}(T), \tag{1.23}$$

where $c(m,n)$ is a constant that depends on m and n.

Their approach was the following: for the proof of the upper bound, they used a dipole construction together with a result concerning the approximation of integral currents by polyhedral ones, and a rather elaborate induction argument. The lower bound of the energy was obtained, in short, in the same manner as in [**4**], through the use of the coarea formula. They presented the proof in much more detail though, giving a variant of the coarea formula that involves the distributional Jacobian, or more generally – for a map taking values in a Riemannian manifold M – the pullback of the volume form on M. This allows them to prove that the current T in (1.22) can be taken to be the slice determined by u at a point $y \in \mathbb{S}^n$ in the subset of Ω where u is approximately differentiable.

In our paper, we will place the problem in the same setting proposed in [**3**], but we discuss the associated weighted energy problem. We will also follow closely their strategy and see that, for the upper bound, replacing the dipole construction in [**3**] with the one originally introduced in [**16**], and being careful at the estimates obtained in the induction process, their method actually yields $c(m,n) = c(n) = n^{n/2}\sigma_{n+1}$ in (1.23), which is exactly the constant from (1.19). Also, inequality (1.22) is still valid for this larger constant, instead of σ_{n+1}.

The problem of the weighted energy was studied, in the classical context, by Millot in *Energy with weight for $\mathbb{S}^2$-valued maps with prescribed singularities* ([**39**]), where he considered the problem (1.18) with energy

$$E_{A_i,d_i}(u,a) := \int_\Omega a(x)|\mathrm{D}u(x)|^2 \,\mathrm{d}x,$$

for a measurable function $a(\cdot)$ that satisfies $0 < \lambda \leqslant a(\cdot) \leqslant \Lambda$, for some given constants λ and Λ. He showed that the formula (1.18) still holds, provided that the distance function used in the expression of the length of the connection is conveniently chosen. He adopted the following definition for the length of a segment (A_1, A_2) in $\mathbb{R}^3$:

$$\ell_a(A_1,A_2) = \liminf_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \int_{\Xi(A_1,A_2;\,\varepsilon)} a(x)\,\mathrm{d}x,$$

where $\Xi(A_1, A_2;\, \varepsilon)$ denotes the set $\{x \in \Omega \mid |x - x_0| \leqslant \varepsilon,$ for some $x_0 \in (A_1, A_2)\}$ obtained by thickening the segment (A_1, A_2). The new distance was then defined by

$$\operatorname{dist}_a(A,B) = \inf \sum_{k=1}^{N} \ell_a(A_k, A_{k+1}), \tag{1.24}$$

where the infimum is taken over all the polygonal lines $(A_1, \ldots, A_{N+1})$ connecting the points A and B, such that dist_1 is the usual Euclidean distance in $\mathbb{R}^3$.

We will consider the case where the function $a(\cdot)$ is continuous, but where, as in [**3**], the dimension and codimension are arbitrary. The exact statement of the main result will be given at the end of the next subsection. As we already mentioned, the strategy used is the one from [**3**]. However, in order to be able to take into consideration the fact that we are placed in a heterogeneous setting, we will have to define a new measure on Ω, by analogy with the distance dist_a defined in [**39**]. Naturally, the area of $\mathcal{S}$ in (1.20) must be replaced with the integral $\displaystyle\int_{\mathcal{S}} a(x)\,\mathrm{d}\mathcal{H}^2(x)$, so we will define a modified Hausdorff measure $\mathcal{H}^h_a$ (which in fact makes sense even if $a(\cdot)$ is only $\mathcal{L}^n$-measurable), and which for $a \equiv 1$ becomes the usual h-dimensional Hausdorff measure on $\mathbb{R}^n$.

1.4.2. Preliminaries and statement of the Theorem. — In this subsection we present the background of the problem. We will consider Ω to be a bounded and smooth open subset of $\mathbb{R}^p$, and M a smooth oriented m-dimensional submanifold of $\mathbb{R}^{n+1}$, with the dimensions satisfying $m \leqslant n+1 \leqslant p$.

1.4.2.1. Rectifiable currents. — We recall here the definitions and the basic properties of the currents with which we are concerned. For further details, see [**50**, Chapter 24], [**47**, Chapter 4], [**30**, Chapter 2, Section 2], and, of course, [**26**].

Basically, a h-dimensional current is a generalization of a distribution, where the place of the test functions $\mathrm{C}^\infty_c(\Omega, \mathbb{R})$ is taken by the h-dimensional differential forms with compact support $\mathcal{D}^h(\Omega)$. More precisely, $\mathcal{D}_h(\Omega)$, the space of h-dimensional currents on Ω, is the topological dual of $\mathcal{D}^h(\Omega)$ with respect to the topology induced by the family of semi-norms $\nu^i_K(\eta) \coloneqq \sup_{j\leqslant i,\, x\in K} \|\mathrm{D}^j\eta(x)\|$ on each $\mathcal{D}^h_K(\Omega) \coloneqq \{\eta \in \mathcal{D}^h(\Omega)\ ;\ \operatorname{supp}\eta \subset K\}$. The vector space $\mathcal{D}_h(\Omega)$ is then endowed with the weak-$\star$ topology, where a sequence $(T_i)_i$ converges to a current T if and only if $T_i(\eta) \to T(\eta)$, $\forall \eta \in \mathcal{D}^h(\Omega)$. The *mass* of T is defined by $\mathbb{M}(T) = \sup_{\|\eta(x)\|\leqslant 1, \forall x} |T(\eta)|$. Here, $\|\eta(x)\|$ denotes the usual Euclidean norm on the space of h-covectors in $\mathbb{R}^p$.

The *boundary* of T is the current $\partial T \in \mathcal{D}_{h-1}(\Omega)$ defined by $\partial T(\mu) \coloneqq T(\mathrm{d}\mu)$, for every $\mu \in \mathcal{D}^{h-1}(\Omega)$, in consistency with Stokes' theorem.

The class of *rectifiable currents* $\mathcal{R}_h(\Omega)$ consists of currents T that can be expressed by an integral formula

$$T(\eta) = \int_{\widetilde{T}} \theta_T(x)\, \langle \eta(x)\,, \tau_T(x)\rangle\, \mathrm{d}\mathcal{H}^h(x), \qquad \forall \eta \in \mathcal{D}^h(\Omega). \tag{1.25}$$

where $\widetilde{T}$ is a compact, h-rectifiable subset of Ω, $\theta_T : \widetilde{T} \to \mathbb{Z}_+$ is a $\mathcal{H}^h$-measurable function called *the multiplicity* of T, and τ_T is an orientation on $\widetilde{T}$. The mass of T then becomes

$$\mathbb{M}(T) = \int_{\widetilde{T}} \theta_T(x)\, \mathrm{d}\mathcal{H}^h(x). \tag{1.26}$$

Given an $\mathcal{H}^h$-integrable function $a(\cdot) : \Omega \to \mathbb{R}$, we can define another current $T \llcorner a$ belonging to $\mathcal{R}^h(\Omega)$, by the formula

$$T \llcorner a(\eta) \coloneqq \int_{\widetilde{T}} a(x)\theta_T(x)\, \langle \eta(x)\,, \tau_T(x)\rangle\, \mathrm{d}\mathcal{H}^h(x), \qquad \forall \eta \in \mathcal{D}^h(\Omega).$$

If $a(\cdot)$ is the characteristic function of a $\mathcal{H}^h$-measurable set A, then $T \llcorner \chi_A$ is also written as $T \llcorner A$, and is called the restriction of T to A.

Given a h-rectifiable set $S \subset \Omega$ oriented by τ_S, we let $[\![S]\!]$ denote the current associated to S through the same expression as in (1.25), where the multiplicity is considered equal to 1. If S is a smooth oriented compact manifold, then, by Stokes' theorem, we have that $\partial[\![S]\!] = [\![\partial S]\!]$.

A current $T \in \mathcal{R}_h(\Omega)$ is called *integral*, and we write $T \in \mathbb{I}_h(\Omega)$, if the boundary ∂T is also rectifiable. By the Closure Theorem [**26**, Theorem 4.2.16], a necessary and sufficient condition for a rectifiable current to be integral is that its boundary has finite mass.

The class of *integral flat currents* $\mathcal{F}_h(\Omega)$ is defined as the set of currents T that can be written as the sum $P + \partial Q$, with $P \in \mathcal{R}_h(\Omega)$, and $Q \in \mathcal{R}_{h+1}(\Omega)$. The flat (semi-)norm on $\mathcal{F}_h(\Omega)$ is defined by

$$\mathcal{F}(T) := \inf \{ \mathbb{M}(P) + \mathbb{M}(Q) \; ; \; T = P + \partial Q, P \in \mathcal{R}_h(\Omega), Q \in \mathcal{R}_{h+1}(\Omega) \}.$$

The space of *integral polyhedral currents* $\mathcal{P}_h(\Omega)$ consists of rectifiable currents that can be written as a finite sum of h-dimensional simplexes with constant integer multiplicities. It represents a dense subset of $\mathcal{F}_h(\Omega)$, with respect to the flat norm. The inclusion relations between the previous classes of currents are the following:

$$\mathcal{P}_h(\Omega) \quad \subset \quad \mathbb{I}_h(\Omega) \quad \subset \quad \mathcal{R}_h(\Omega) \quad \subset \quad \mathcal{F}_h(\Omega) \quad \subset \quad \mathcal{D}_h(\Omega).$$

1.4.2.2. The pullback $u^\sharp\omega$ of the volume form of an oriented manifold. — Denote by ω the (standard) volume form on M, that is, the differential m-form on M which associates to each point $y \in M$ the m-linear alternating map on $\mathrm{T}_y M$ that satisfies

$$\omega(y)(\tau_1, \dots, \tau_m) = 1,$$

for any $\{\tau_1, \dots, \tau_m\}$ positively oriented orthonormal basis of $\mathrm{T}_y M$. Since the top-dimensional forms constitute a 1-dimensional vector space, we see that ω is necessarily given by

$$\omega(y)(w_1, \dots, w_m) = \det (w_1, \dots, w_m, \nu_1, \dots, \nu_{n-m+1}) \, ,$$

for every $y \in M$, $w_1, \dots, w_m \in \mathrm{T}_y M$, and any $\{\nu_1, \dots, \nu_{n-m+1}\} \subset \mathbb{R}^{n+1}$ a positively oriented orthonormal basis of $\mathrm{N}_y M = \mathrm{T}_y M^\perp$. The value of the above determinant is independent of the choice of basis of the normal space since passing to another one leads to the multiplication of the matrix in the right-hand side by an orthogonal matrix of determinant equal to 1.

Suppose $u : \Omega \to M$ is a differentiable map. We can then consider the pullback of the volume form ω with respect to u, which is the m-form on Ω defined by

$$(u^\sharp\omega)(x)(v_1, \dots, v_m) = \omega(u(x))(\mathrm{d}u_x(v_1), \dots, \mathrm{d}u_x(v_m)),$$

for every $x \in \Omega$ and $v_1, \dots, v_m \in \mathbb{R}^p$. Remark that in [**3**], the notation $\mathrm{J}_\omega u$ is used instead of $u^\sharp\omega$, to emphasise the role that it plays in the definition of the distributional Jacobian.

For every $x \in \Omega$, the form $(u^\sharp\omega)(x)$ is an element of the space $\Lambda^m(\mathbb{R}^p)$ of the m-covectors in $\mathbb{R}^p$, which can be endowed with the Euclidean norm, meaning that if we let $\mathrm{I}(m,p)$ denote the set of ordered multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$ with $1 \leqslant \alpha_1 < \dots < \alpha_m \leqslant p$, and $\{\mathrm{d}x_1, \dots, \mathrm{d}x_p\}$ is the dual base of an orthonormal basis $\{e_1, \dots, e_p\}$ of $\mathbb{R}^p$, then

$$(u^\sharp\omega)(x) = \sum_{\alpha \in \mathrm{I}(m,p)} (u^\sharp\omega)(x)(e_{\alpha_1}, \dots, e_{\alpha_m}) \, \mathrm{d}x_{\alpha_1} \wedge \dots \wedge \mathrm{d}x_{\alpha_m},$$

and

$$|(u^\sharp\omega)(x)|^2 = \sum_{\alpha \in \mathrm{I}(m,p)} |(u^\sharp\omega)(x)(e_{\alpha_1}, \dots, e_{\alpha_m})|^2.$$

With this, we can see that $|(u^\sharp\omega)(x)|$ is simply the Jacobian $[\![\mathrm{D}u(x)]\!]$ of the $m \times (n+1)$-matrix $\mathrm{D}u(x)$ of the differential of u at x, that appears in the classical coarea formula, that is,

$$|(u^\sharp\omega)(x)| = [\![\mathrm{D}u(x)]\!] = \left[\det \left(\mathrm{D}u(x) \, \mathrm{D}u(x)^\star\right)\right]^{1/2}.$$

Indeed, writing the matrix of the differential of u in x with respect to the canonical basis $\{e_1,\dots,e_p\}$ in $\mathbb{R}^p$ and an orthonormal basis $\{\tau_1,\dots,\tau_m\}$ in $\mathrm{T}_{u(x)}M$,

$$\mathrm{d}u_x(e_i)=\sum_{j=1}^{m}\lambda_{ji}\tau_j=\lambda_i\cdot(\tau_1,\dots,\tau_m),\quad \forall i=1,\dots,p,$$

clearly gives us

$$|(u^\sharp\omega)(x)|^2=\sum_{\alpha\in\mathrm{I}(m,p)}|\det(\lambda_{\alpha_1},\dots,\lambda_{\alpha_m})|^2=[\![\mathrm{D}u(x)]\!]^2.$$

Remark that the quantity $|(u^\sharp\omega)(x)|$ is called *the m-dimensional Jacobian of u at x* in [**4**]. We will later make use of the following estimate:

(1.27) $\quad |(u^\sharp\omega)(x)|\leqslant m^{-m/2}\times|\mathrm{D}u(x)|^m,\quad$ for all $x\in\Omega$,

that can be found also in [**4**, Appendix A.1.3]. In fact, it was obtained before that in [**16**, (8.5)] in the case where $M\equiv\mathbb{S}^m$. In that context, we have $|(u^\sharp\omega)(x)|=|D|$, with D representing the vector field $(D_1,\dots,D_p)$ that is defined by

$$D_j=\det(\partial_1u,\dots,\partial_{j-1}u,u,\partial_{j+1}u,\dots,\partial_pu),\quad j\in[\![1,\dots,p]\!].$$

To verify inequality (1.27), suppose first that u is a submersion at the point $x\in\Omega$, that is, the differential

$$\mathrm{d}u_x:\mathbb{R}^p\to\mathrm{T}_{u(x)}M$$

is surjective and so its null space has dimension $(p-m)$. If we choose $\{e_1,\dots,e_p\}$ to be an orthonormal basis in $\mathbb{R}^p$ such that $\ker(\mathrm{d}u_x)=\mathrm{span}\{e_{m+1},\dots,e_p\}$, then only one term in the expression of $|(u^\sharp\omega)(x)|$ is non zero, namely the one indexed by $\alpha=(1,\dots,m)$, hence

$$|(u^\sharp\omega)(x)|=\left|\det\left(\mathrm{d}u_x(e_1),\dots,\mathrm{d}u_x(e_m),\nu_1,\dots,\nu_{n-m+1}\right)\right|,$$

where $\{\nu_1,\dots,\nu_{n-m+1}\}$ is an orthonormal basis in $\mathrm{N}_{u(x)}M$. By Hadamard's inequality and the arithmetic-geometric mean inequality we obtain

$$\begin{aligned}|(u^\sharp\omega)(x)|^2&\leqslant|\mathrm{d}u_x(e_1)|^2\dots|\mathrm{d}u_x(e_m)|^2\\&\leqslant\left[\frac{1}{m}\left(|\mathrm{d}u_x(e_1)|^2+\cdots+|\mathrm{d}u_x(e_m)|^2\right)\right]^m=\left(\frac{|\mathrm{D}u(x)|^2}{m}\right)^m.\end{aligned}$$

If however, $\mathrm{d}u_x$ is not surjective, then any m vectors $\mathrm{d}u_x(e_{\alpha_1}),\dots,\mathrm{d}u_x(e_{\alpha_m})$ are linearly dependent, and so, $|(u^\sharp\omega)(x)|$ is zero.

Notice that everything remains true almost everywhere if we suppose $u:\Omega\to M$ is only Lipschitz. Also, the definition of the pullback of ω makes sense if u is merely in $\mathrm{W}^{1,m}(\Omega,M)$, and then $|(u^\sharp\omega)(\cdot)|$ belongs to $\mathrm{L}^1(\Omega)$.

Rather than the form $u^\sharp\omega$, the associated current $\star u^\sharp\omega$ will be of more interest in what follows. The action of the Hodge-type operator $\star$ on $u^\sharp\omega$ is given by

$$\star(u^\sharp\omega)(x)=\sum_{\alpha\in\mathrm{I}(p-m,p)}\mathrm{sgn}(\alpha,\bar\alpha)(u^\sharp\omega)(x)(e_{\bar\alpha_1}\dots,e_{\bar\alpha_m})\,e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_{p-m}},$$

which thus becomes a $(p-m)$-current on Ω. Here, $\bar\beta$ denotes the multi-index in $\mathrm{I}(m,p)$ that complements the element $\beta\in\mathrm{I}(p-m,p)$.

1.4.2.3. An integral representation of the current $\star u^\sharp\omega$ via the coarea formula. — Suppose $u : \Omega \to M$ is a Lipschitz map. The coarea formula (see [**25**, Theorem 3.1], [**24**, Section 3.4]) states that given an L^p-integrable function $a(\cdot) : \Omega \to \mathbb{R}$, and an L^p-measurable set $A \subset \mathbb{R}^p$, we have that, for $\mathcal{H}^m$-almost every $y \in M$, the level set $u^{-1}(y)$ is $\mathcal{H}^{p-m}$-rectifiable, and

$$\int_A a(x)[\![\mathrm{D}u(x)]\!]\,\mathrm{d}x = \int_M \left(\int_{u^{-1}(y)\cap A} a(x)\,\mathrm{d}\mathcal{H}^{p-m}(x)\right)\mathrm{d}\mathcal{H}^m(y), \tag{1.28}$$

hence, also, for any continuous function $\rho : M \to \mathbb{R}$,

$$\int_A \rho(u(x))[\![\mathrm{D}u(x)]\!]\,\mathrm{d}x = \int_M \rho(y)\mathcal{H}^{p-m}\left(u^{-1}(y)\cap A\right)\mathrm{d}\mathcal{H}^m(y), \tag{1.29}$$

assuming that the left-hand side of (1.29) is integrable.

If we choose the set A conveniently, we see that for $\mathcal{H}^{p-m}$-almost every y in M, the Jacobian $[\![\mathrm{D}u(x)]\!] = |u^\sharp\omega(x)|$ is defined and is non zero, for $\mathcal{H}^{p-m}$-almost any x in $u^{-1}(y)$. At every such points x, the differential du_x vanishes on the tangent space of the level set $u^{-1}(y)$, and so, using a basis $\{e_1,\dots,e_p\}$ of $\mathbb{R}^p$ for which $\mathrm{Tan}_x u^{-1}(y) = \mathrm{span}\{e_1,\dots,e_{p-m}\}$, we have

$$\star u^\sharp\omega(x) = \omega(u(x))(\mathrm{d}u_x(e_{p-m+1}),\dots,\mathrm{d}u_x(e_p))\,e_1\wedge\cdots\wedge e_{p-m},$$

hence, after normalisation, $\star u^\sharp\omega$ defines an orientation of the rectifiable set $u^{-1}(y)$. Thus, $u^{-1}(y)$ becomes a rectifiable current of multiplicity equal to 1, that is,

$$[\![u^{-1}(y)]\!](\eta) = \int_{u^{-1}(y)} \left\langle \eta(x), \frac{\star u^\sharp\omega(x)}{|\star u^\sharp\omega(x)|}\right\rangle \mathrm{d}\mathcal{H}^{p-m}(x),$$

for all $\eta \in \mathrm{D}^{p-m}(\Omega)$, and for $\mathcal{H}^m$-almost every $y \in M$. Equivalently, $[\![u^{-1}(y)]\!]$ is the $\mathbb{R}^{p-m}$-valued measure $\dfrac{\star u^\sharp\omega(x)}{|\star u^\sharp\omega(x)|}\mathcal{H}^{p-m} \llcorner \{u^{-1}(y)\}$, so its total variation is

$$|[\![u^{-1}(y)]\!]| = \mathcal{H}^{p-m} \llcorner \{u^{-1}(y)\},$$

meaning that for all $\varphi \in \mathrm{C_c}(\Omega)$, we have

$$|[\![u^{-1}(y)]\!]|(\varphi) = \int_{u^{-1}(y)} \varphi(x)\,\mathrm{d}\mathcal{H}^{p-m}(x).$$

By approximating φ with simple functions, we see that

$$\int_{u^{-1}(y)} \varphi(x)\,\mathrm{d}\mathcal{H}^{p-m}(x) \quad \text{is } \mathcal{H}^m\text{-measurable}$$

(as pointwise limit of $\mathcal{H}^m$-measurable functions), that is, the mapping $y \mapsto |[\![u^{-1}(y)]\!]|$ is weak-$\star$-measurable. Applying the coarea formula and the monotone convergence theorem, we obtain that, for any $\varphi \in \mathrm{C_c}(\Omega)$,

$$\int_\Omega |\rho(u(x))|\cdot|u^\sharp\omega(x)|\,\varphi(x)\,\mathrm{d}x = \int_M |\rho(y)|\cdot|[\![u^{-1}(y)]\!]|(\varphi)\,\mathrm{d}\mathcal{H}^m(y). \tag{1.30}$$

The integrals are finite, because we have

$$\begin{aligned}\int_M |\rho(y)|\cdot\big|[\![u^{-1}(y)]\!](\varphi)\big|\ \mathrm{d}\mathcal{H}^m(y) &\leqslant \sup_\Omega|\varphi|\int_M |\rho(y)|\,\mathcal{H}^{p-m}(u^{-1}(y))\ \mathrm{d}\mathcal{H}^m(y)\\ &= \sup_\Omega|\varphi|\int_\Omega |\rho(u(x))|\cdot|u^\sharp\omega(x)|\ \mathrm{d}x\\ &\leqslant m^{-m/2}\sup_\Omega|\varphi|\sup_M|\rho|\int_\Omega|\mathrm{D}u(x)|^m\ \mathrm{d}x,\end{aligned}$$

which is finite, since Ω is bounded, and thus, the map $|[\![u^{-1}(\cdot)]\!]|$ is weak-$\star$- integrable. The same can be said about the function $|u^\sharp\omega(\cdot)|$, as it belongs to $\mathrm{L}^1(\Omega)$, and so, we can express the equality (1.30) in short as

$$|u^\sharp(\rho\omega)| = \int_M^\star |\rho(y)|\cdot|[\![u^{-1}(y)]\!]|\ \mathrm{d}\mathcal{H}^m(y),$$

the integral sign $\int^\star$ meaning that it is a weak-$\star$ (also known as Gelfand) integral. Furthermore, since the identity (1.30) is in fact true for every bounded function $\varphi\in\mathrm{L}^1(\Omega)$, we can take φ as defined by the duality product

$$\varphi(x) = \left\langle \star u^\sharp(\rho\omega)(x), \frac{\eta(x)}{|u^\sharp(\rho\omega)(x)|}\right\rangle,$$

for any $\eta\in\mathcal{D}^{p-m}(\Omega)$, and we get

$$\begin{aligned}&\int_\Omega\langle\star u^\sharp(\rho\omega)(x),\eta(x)\rangle\ \mathrm{d}x\\ &\qquad= \int_M\rho(y)\left[\int_{u^{-1}(y)}\left\langle\star u^\sharp\omega(x),\frac{\eta(x)}{|u^\sharp\omega(x)|}\right\rangle\mathrm{d}\mathcal{H}^{p-m}(x)\right]\mathrm{d}\mathcal{H}^m(y).\end{aligned}$$

This can be also written as

$$\star u^\sharp(\rho\omega) = \int_M^\star \rho(y)[\![u^{-1}(y)]\!]\ \mathrm{d}\mathcal{H}^m(y). \tag{1.31}$$

Suppose now that the map u belongs to $\mathrm{W}^{1,1}(\Omega,M)$. In order to deduce an analogue of the coarea formula for Sobolev functions from the classical one, it is natural to try to use some Luzin type approximation results. We begin by covering Ω, up to a Lebesgue null set $E\subset\Omega$, by a disjoint sequence Ω_j of measurable subsets in $\mathbb{R}^p$ with the property that the restriction u_j of u to every such set is Lipschitz (using for example [**24**, Theorem 3, Section 6.6.3]). We can then apply formula (1.28) to every $u_j:\Omega_j\to M$, to obtain

$$\int_{A\cap\Omega_j}a(x)|u^\sharp\omega(x)|\ \mathrm{d}x = \int_M\left(\int_{u_j^{-1}(y)\cap\Omega_j\cap A}a(x)\ \mathrm{d}\mathcal{H}^{p-m}(x)\right)\mathrm{d}\mathcal{H}^m(y),$$

for any $\mathcal{L}^p$-measurable set $A\subset\Omega$ and $\mathcal{L}^p$-integrable function $a(\cdot):\Omega\to\mathbb{R}$. Taking the sum over all j, we get,

$$\int_A a(x)|u^\sharp\omega(x)|\ \mathrm{d}x = \int_M\left(\int_{u^{-1}(y)\smallsetminus E\cap A}a(x)\ \mathrm{d}\mathcal{H}^{p-m}(x)\right)\mathrm{d}\mathcal{H}^m(y),$$

at least when $a(\cdot)$ is nonnegative, where, in the left-hand, we used the fact that the set E has Lebesgue measure zero. However, since the restriction $u|_E$ is not Lipschitz, it need not have the Luzin N-property, hence $\mathcal{H}^{p-m}(u^{-1}(y)\cap A\cap E)$ is not necessarily equal to zero, so in the analogue of the coarea formula for u we have to keep in mind that we have to remove the set E from the usual level set $u^{-1}(y)$. What we can say about this set E is that the function u is almost everywhere approximately differentiable on its complement $\Omega \smallsetminus E$, since every Lipschitz function is approximately differentiable almost everywhere. But in fact, as presented in [**3**], in view of [**26**, Thm.3.1.8] (see also [**30**, Theorem 3, Section 3.1.4]), we can take, in the above, E to be exactly the set of the points of non-approximate differentiability of u. By using the notation $N_y \coloneqq u^{-1}(y) \smallsetminus E$, we have thus arrived to the following coarea formula

$$\int_A a(x)|u^\sharp\omega(x)|\,\mathrm{d}x = \int_M \left(\int_{N_y\cap A} a(x)\,\mathrm{d}\mathcal{H}^{p-m}(x)\right)\mathrm{d}\mathcal{H}^m(y), \tag{1.32}$$

and also, by the same argument as before, viewing N_y as a rectifiable current of constant multiplicity 1, we deduce the representation formula

$$\star u^\sharp(\rho\omega) = \int_M^\star \rho(y)[\![N_y]\!]\,\mathrm{d}\mathcal{H}^m(y), \tag{1.33}$$

where the maps $a(\cdot)$ and ρ satisfy the same properties as before.

1.4.2.4. The distributional Jacobian. — If u is a bounded map in $\mathrm{W}^{1,n}(\Omega,\mathbb{R}^{n+1})$, the distributional Jacobian of u is defined as the $(n+1)$-form on Ω given, in the distributional sense, by

$$\mathrm{J}u \coloneqq \frac{1}{n+1}\,\mathrm{d}(u^\sharp\omega_0), \tag{1.34}$$

where ω_0 represents the volume form of the sphere $\mathbb{S}^n$. Since we have

$$\omega_0(y) = \sum_{i=1}^{n+1}(-1)^{i-1}y_i\,\widehat{\mathrm{d}y_i}, \quad \text{for every } y\in\mathbb{S}^n,$$

where $\widehat{\mathrm{d}y_i} \equiv \mathrm{d}y_1\wedge\ldots\mathrm{d}y_{i-1}\wedge\mathrm{d}y_{i+1}\wedge\ldots\mathrm{d}y_{n+1}$, it means that

$$\mathrm{J}u = \frac{1}{n+1}\,\mathrm{d}\left(\sum_{\alpha\in I(n,p)}\det\left[\partial_{\alpha_1}u,\ldots,\partial_{\alpha_n}u,u\right]\,\mathrm{d}x_{\alpha_1}\wedge\cdots\wedge\mathrm{d}x_{\alpha_n}\right)$$

The paper [**2**] presents a review of the distributional Jacobian. We recall here a couple of basic properties of $\mathrm{J}u$. First of all, if u is more regular, say it belongs to $\mathrm{W}^{1,n+1}(\Omega,\mathbb{R}^{n+1})$, then

$$\mathrm{J}u = \sum_{\alpha\in I(n+1,p)}\det\left[\partial_{\alpha_1}u,\ldots,\partial_{\alpha_{n+1}}u\right]\,\mathrm{d}x_{\alpha_1}\wedge\cdots\wedge\mathrm{d}x_{\alpha_{n+1}},$$

so, if $p=n+1$, then $\mathrm{J}u$ is simply $\det(\mathrm{D}u)\,\mathrm{d}x$. This explains the choice of the factor $\dfrac{1}{n+1}$ in the definition. Secondly, if $u\in\mathrm{W}^{1,n+1}(\Omega,\mathbb{S}^n)$, then the partial derivatives of u being pointwise linearly dependent, we necessarily have $\mathrm{J}u=0$.

A useful remark, given by [**3**, Prop.7.9], is that, in the identity (1.34), we can replace ω_0 by any differential n-form $\eta\coloneqq\rho\omega_0$ with $\rho:\mathbb{S}^n\to\mathbb{R}$ a smooth function with average equal to 1. Indeed, the integral over $\mathbb{S}^n$ of the n-form $(1-\rho)\omega_0$ equals zero, which implies that it is an exact form on $\mathbb{S}^n$ (see for example [**34**, Theorem VII.B.6]). Then the conclusion follows, since the exterior derivative d commutes with the pullback and $\mathrm{d}\circ\mathrm{d}=0$.

The distributional Jacobian $\mathrm{J}u$ describes in some sense the topological singularities of the map u. To exemplify, concerning $u \in \mathrm{W}^{1,n}(\Omega \subset \mathbb{R}^p, \mathbb{S}^n)$, the following are true:

(i) In the case $p = n+1$, the map u can be approximated, in the $\mathrm{W}^{1,n}$ norm, by a sequence $(u_j)_j \subset \mathrm{C}^\infty(\Omega, \mathbb{S}^n)$ if and only if $\mathrm{J}u = 0$.

(ii) There exists a sequence $(u_j)_j \subset \mathrm{C}^\infty(\Omega \setminus \Gamma_j, \mathbb{S}^n)$ such that $u_j \to u$ in $\mathrm{W}^{1,n}$, where $\Gamma_j = \partial(u_j^{-1}(y))$, for $\mathcal{H}^n$-almost every $y \in \mathbb{S}^n$.

(iii) If the mapping u is continuous outside a $(p-n-1)$-dimensional submanifold S, then $\star \mathrm{J}u = \dfrac{\sigma_{n+1}}{n+1} \deg(u, S)[\![S]\!]$.

(iv) In general, $\star \mathrm{J}u = (-1)^{p-n} \dfrac{\sigma_{n+1}}{n+1} \partial [\![N_y]\!]$ for $\mathcal{H}^n$-almost all $y \in \mathbb{S}^n$.

The property (i) is the main result of [**7**] (see Remark 2 therein). The approximation result (ii) can be proved by following the same argument used in [**14**, Section 5] to prove the density of the class $\mathcal{R}$ in $\mathrm{W}^{s,q}(\mathbb{S}^N, \mathbb{S}^1)$ (for $s, q \geqslant 1$ with $1 \leqslant sq < 2$, [**14**, Theorem 2]). The property (iii) is stated in [**3**, Subsection 3.7] and it was previously proved, for $p = n+1$ in [**16**, Theorem B.2, Remark B.2], for $n = 2$ and S a $(p-3)$-dimensional disk in [**49**, Corollary 1], and for $p = 3$, $n = 1$ in [**33**, Example 3.4]. And finally, (iv) is one of the conclusions of [**3**, Theorem 3.8], and we detail below its proof.

We have seen that the Jacobian of u is given by $\mathrm{J}u = \dfrac{1}{n+1} \mathrm{d}(u^\sharp(\rho\omega_0))$, where ρ is any real smooth function on $\mathbb{S}^n$ with average equal to 1, and ω_0 is the volume form on $\mathbb{S}^n$. Hence, by the properties of the $\star$ operator, we have

$$\star \mathrm{J}u = \frac{(-1)^m}{n+1} \partial(\star u^\sharp(\rho\omega_0)). \tag{1.35}$$

We fix a point $z \in \mathbb{S}^n$ where $y \mapsto [\![N_y]\!]$ is weak-$\star$ approximately continuous. This is in fact true for $\mathcal{H}^n$-almost all z, because, as we have seen, the map $y \mapsto [\![N_y]\!]$ is weak-$\star$ $\mathcal{H}^n$-measurable. So, if we consider $\{\eta_i\}$ a dense countable subset of $\mathcal{D}^m(\Omega)$, then, for every $\eta \in \mathcal{D}^m(\Omega)$, $y \mapsto [\![N_y]\!](\eta)$ is $\mathcal{H}^n$-measurable, hence $\mathcal{H}^n$-almost everywhere approximately continuous. Thus, for $\mathcal{H}^n$ almost all z, the map $y \mapsto [\![N_y]\!](\eta_i)$ is approximately continuous in z, for every i, and, by density, it follows that all such points z are actually points of weak-$\star$ approximate continuity of $[\![N_y]\!]$.

Let $\rho_i : \mathbb{S}^n \to \mathbb{R}$ be a sequence of positive smooth functions with compact support which approximate the Dirac measure at z, such that

$$\operatorname{supp}(\rho_i) = \overline{B(z, r_i)} \cap \mathbb{S}^n, \quad \int_{\mathbb{S}^n} \rho_i(x)\, \mathrm{d}\mathcal{H}^n(x) = 1, \quad \text{and}$$
$$\sup_i \sup_x \{ |\rho_i(x)|\, \mathcal{H}^n(\operatorname{supp}(\rho_i)) \} =: \alpha < \infty.$$

Then, using the formula (1.33), we have that, for any $\eta \in \mathcal{D}^m(\Omega)$,

$$\begin{aligned} |\star u^\sharp(\rho_i\omega_0)(\eta) - [\![N_z]\!](\eta)| &= \left| \int_{\mathbb{S}^n} \rho_i(y) [\![N_y]\!](\eta)\, \mathrm{d}\mathcal{H}^n(y) - [\![N_z]\!](\eta) \right| \\ &\leqslant \int_{\mathbb{S}^n} \rho_i(y)\, |[\![N_y]\!](\eta) - [\![N_z]\!](\eta)|\, \mathrm{d}\mathcal{H}^n(y) \\ &\leqslant \alpha \fint_{B(z,r_i)\cap\mathbb{S}^n} |[\![N_y]\!](\eta) - [\![N_z]\!](\eta)|\, \mathrm{d}\mathcal{H}^n(y), \end{aligned}$$

which converges to 0 when r_i tends to 0, hence $\star u^\sharp(\rho_i\omega_0)$ converges, in the weak-$\star$ topology, to $[\![N_z]\!]$. Since the boundary operator is continuous, this implies that

$$\partial\,[\![N_z]\!] = \frac{(-1)^m(n+1)}{\sigma_{n+1}} \star \mathrm{J}u, \tag{1.36}$$

noting that we have to use $\sigma_{n+1}\rho_i$ in place of ρ in the relation (1.35).

We can now state the main result of the paper.

Theorem 1.1. — *Let Ω be a bounded and connected smooth open set in $\mathbb{R}^{m+n}$, and suppose Γ is the boundary of a rectifiable current in Ω, of dimension m and finite mass. Consider the following class of admissible functions*

$$\mathcal{E}(\Gamma) \coloneqq \left\{ u \in \mathrm{W}^{1,n}(\Omega, \mathbb{S}^n) \;\middle|\; \star\mathrm{J}u = \frac{\sigma_{n+1}}{n+1}\,\Gamma \right\},$$

To each map u in $\mathcal{E}(\Gamma)$, we associate the weighted energy

$$E_\Gamma(u,a) = \int_\Omega a(x)|\mathrm{D}u(x)|^n \,\mathrm{d}x,$$

for any strictly positive, continuous function $a(\cdot)$ on Ω, with $\inf a(\cdot) > 0$. *Then, the minimum energy of u is given by*

$$\inf_{u\in\mathcal{E}(\Gamma)} E_\Gamma(u,a) = n^{n/2}\sigma_{n+1} \inf_{M\in\mathcal{C}(\Gamma)} \mathbb{M}(M \llcorner a), \tag{1.37}$$

where the infimum in the right-hand side is taken over the following class of integral currents,

$$\mathcal{C}(\Gamma) \coloneqq \{ M \in \mathcal{R}_m(\mathbb{R}^{m+n}) \mid \mathbb{M}(M) < \infty,\ \partial M = \Gamma \}.$$

1.4.3. The proof of Theorem 1.1.— The first step is the construction of the dipole which will be used in the proof of the upper bound of the energy. This intermediate result, given in Lemma 1.5, follows the one in [**16**, Section VIII, 1. The Upper Bound], concerned with the case $m = 2$, $n = 1$. In order to take into account a continuous weight function, we define the following modified spherical Hausdorff measure.

Definition 1.2. — For any positive $\mathcal{L}^{p/m}_{\mathrm{loc}}$-measurable function $w(\cdot)$ on $\mathbb{R}^p$, let $\mathcal{H}^m_w$ denote the outer measure on $\mathbb{R}^p$ defined by

$$\mathcal{H}^m_w(A) = \sup_{\delta>0} \inf \left\{ \sum_{k=1}^N \frac{p^{m/p}\sigma_m}{m\sigma_p^{m/p}} \|w\|_{\mathrm{L}^{p/m}(D_k)} \;\middle|\; \begin{array}{r} D_k \subset \mathbb{R}^p \text{ open balls that cover } A \\ \operatorname{diam} D_k < \delta \end{array} \right\},$$

for every $A \subset \mathbb{R}^p$.

Since $\mathcal{H}^m_w$ is obtained, like the usual Hausdorff measure, *via* the Carathéodory construction, it is indeed a (metric) outer measure on $\mathbb{R}^p$, and the following result shows the connection between $\mathcal{H}^m_w$ and $\mathcal{H}^m$.

Proposition 1.3. — *If $w(\cdot)$ is a continuous positive function, then*

$$\mathcal{H}^m_w \llcorner A = w(\cdot)\mathcal{H}^m \llcorner A,$$

for any $\mathcal{H}^m$-rectifiable set $A \subset \mathbb{R}^p$.

Proof. — First we see that, if $w(\cdot)$ is constant, then the above equality is satisfied, because, by [**26**, Theorem 3.2.26], the Hausdorff and spherical measures coincide on $\mathcal{H}^m$-rectifiable sets. Next, remark that the two measures, $\mathcal{H}^m_w \llcorner A$ and $w(\cdot)\mathcal{H}^m \llcorner A$, are both Radon measures since, by hypothesis, $\mathcal{H}^m(A) < \infty$, and so, also $\mathcal{H}^m_w(A) < \infty$. The sets of null $\mathcal{H}^m_w$ and $\mathcal{H}^m$-measure coincide, so it is enough to prove

$$\mathcal{H}^m_w(A) = \int_A w(x)\,\mathrm{d}\mathcal{H}^m(x),$$

for any set A that is Borel $\mathcal{H}^m$-rectifiable and compact.

We will apply the Vitali-Besicovitch covering theorem (see [**5**, Theorem 2.19]), to cover A, up to a set of small $\mathcal{H}^m$-measure, with a disjoint family of small balls. For this, first, for any $\varepsilon > 0$, we choose $\gamma > 0$ so that $w(\cdot)$, which is uniformly continuous on A, satisfies $|w(x) - w(y)| \leqslant \varepsilon$, whenever $|x - y| \leqslant \gamma$. Then, for any $\delta > 0$, we can find a finite number of balls such that

$$\mathcal{H}^m\left(A \smallsetminus \bigcup_{j=1}^N B_{\gamma_j}(x_j)\right) \leqslant \delta\,, \quad \text{where} \quad \gamma_j \leqslant \gamma,\ x_j \in A,\ \forall j = 1,\dots,N.$$

With this, we have

$$\int_A w(x)\,\mathrm{d}\mathcal{H}^m(x) \leqslant \delta \sup_A w(\cdot) + \varepsilon\mathcal{H}^m(A) + \sum_{j=1}^N w(x_j)\mathcal{H}^m(B_{\gamma_j}(x_j) \cap A)$$

and, knowing that

$$w(x_j)\mathcal{H}^m \llcorner A = \mathcal{H}^m_{w(x_j)} \llcorner A, \quad \text{for all } j, \quad \text{and}$$
$$w(x_j) \leqslant \varepsilon + w(x), \quad \text{for all } x \in B_{\gamma_j}(x_j),$$

we arrive to

$$\int_A w(x)\,\mathrm{d}\mathcal{H}^m(x) \leqslant \delta \sup_A w(\cdot) + 2\varepsilon\mathcal{H}^m(A) + \mathcal{H}^m_w(A).$$

On the other hand, we get, in the same manner, that

$$\mathcal{H}^m_w(A) \leqslant \delta \sup_A w(\cdot) + 2\varepsilon\mathcal{H}^m(A) + \int_A w(x)\,\mathrm{d}\mathcal{H}^m(x),$$

so, by taking ε, $\delta \to 0$, we have obtained the desired equality. □

We assume, from now on, the hypotheses of Theorem 1.1. Following the previous lemma, we could define a new mass $\mathbb{M}_a$ on $\mathcal{R}_m(\mathbb{R}^{m+n})$ by substituting $\mathcal{H}^m$ for $\mathcal{H}^m_a$ in the integral representation of the mass $\mathbb{M}$, see (1.26) in Subsection 1.4.2. Specifically, for a current $T \in \mathcal{R}_m(\mathbb{R}^{m+n})$ associated to the $\mathcal{H}^m$-rectifiable set $\widetilde{T}$ and the multiplicity θ_T, we set

$$\mathbb{M}_a(T) \coloneqq \int_{\widetilde{T}} \theta_T(x)\mathcal{H}^m_a.$$

With this, formula (1.37) can be written as

$$\inf_{u\in\mathcal{E}(\Gamma)} E_\Gamma(u,a) = n^{n/2}\sigma_{n+1} \inf_{M\in\mathcal{C}(\Gamma)} \mathbb{M}_a(M),$$

which is in agreement with the observation in [**16**] that different generalizations of the original problem should all yield the same kind of formula for the minimum energy, provided that the distance, in this case the measure, is properly defined. The same approach was taken in [**39**], where the Euclidean distance in $\mathbb{R}^3$ was replaced by dist_a

whose definition we specified in the Introduction, formula (1.24). Considering the proposition below, it is clear that Theorem 1.1 gives, for the case $m = 1$, $n = 2$, the same formula, as does [**39**, Theorem 1.1], for $a(\cdot)$ continuous.

Proposition 1.4. — *Suppose* Γ *is a polyhedral current. Then, in the formula* (1.37) *of Theorem 1.1, we can assume that the currents* M *are polyhedral, i.e.,*

$$\inf\{\mathbb{M}(M \llcorner a) \mid M \in \mathcal{C}(\Gamma)\} = \inf\{\mathbb{M}(M \llcorner a) \mid M \in \mathcal{C}(\Gamma) \cap \mathcal{P}_m(\mathbb{R}^{m+n})\}\,.$$

Proof. — We need to show that, given $M \in \mathcal{R}_m(\mathbb{R}^{m+n})$, with finite mass and boundary $\partial M = \Gamma$, we can find, for every $\varepsilon > 0$, a polyhedral current P, that has the same boundary, and which satisfies

$$\mathbb{M}(P \llcorner a) \leqslant \mathbb{M}(M \llcorner a) + \varepsilon.$$

It suffices to show this for $a(\cdot) \equiv 1$, as the general case follows by applying the Vitali-Besicovitch theorem as in the precedent proof.

Let $\eta > 0$. By the approximation theorem [**26**, Theorem 4.2.22], there exists a polyhedral current $P_0 \in \mathcal{P}_m(\mathbb{R}^{m+n})$, and two rectifiable currents R and Q, of dimension m and $m+1$, respectively, such that

$$M = P_0 + R + \partial Q, \qquad \text{with} \qquad \mathbb{M}(R) \leqslant \eta, \quad \text{and} \quad \mathbb{M}(P_0) \leqslant \mathbb{M}(M) + \eta.$$

We may assume $0 < \eta < \varepsilon/2$ is small enough so that, using the deformation theorem [**26**, Theorem 4.2.9] we can write R as the sum $P_1 + \partial S$, between a polyhedral current P_1 satisfying

$$\mathbb{M}(P_1) \leqslant \frac{\varepsilon}{2}$$

and the boundary ∂S of a $(m+1)$-dimensional integral current S. Therefore, by taking $P := P_0 + P_1$, we have

$$M = P + \partial(Q + S), \qquad \text{with} \qquad \mathbb{M}(P) \leqslant \mathbb{M}(P_0) + \mathbb{M}(P_1) \leqslant \mathbb{M}(M) + \varepsilon,$$

and $\partial P = \partial M = \Gamma$. □

The following lemma gives the analogue of the dipole introduced in [**16**].

Lemma 1.5. — *Suppose* $E \equiv E_r(\xi_0, \tau_E)$ *is an oriented disk of dimension* m *in* Ω. *Then, for any* $\delta > 0$, *and any fixed point* $y_0 \in \mathbb{S}^n$, *there exists a map* $u \in \mathcal{E}([\![\partial E]\!])$, *that is locally Lipschitz in* $\Omega \smallsetminus \partial E$, *constantly equal to* y_0 *outside the ball* $B_r(\xi_0)$ *in* $\mathbb{R}^{m+n}$, *and whose energy satisfies the following inequality*

$$\int_\Omega a(x)\,|\mathrm{D}u(x)|^n\,\mathrm{d}x \leqslant n^{n/2}\sigma_{n+1}\mathcal{H}^m_a(E) + \delta. \tag{1.38}$$

Proof. — Without loss of generality, we may restrict to the particular case where

$$E = \{(x,0) \in \mathbb{R}^m \times \mathbb{R}^n \text{ with } |x| \leqslant r\},$$
$$\tau_E = e_1^{(m+n)} \wedge \cdots \wedge e_m^{(m+n)} \in \mathbb{R}^m \times \mathbb{R}^n, \qquad \text{and} \qquad y_0 = (0,1) \in \mathbb{R}^n \times \mathbb{R},$$

because the general case will then follow through an isometry of $\mathbb{R}^{n+m}$, and a rotation of $\mathbb{S}^n$.

The map u will be constructed in two steps, just as described in [**16**]. The first step is to take a function $v : \mathbb{R}^n \to \mathbb{S}^n$ that satisfies the following properties:

$$v(\cdot) \text{ is Lipschitz on } \mathbb{R}^n \text{ and constant outside a small ball,} \tag{1.39}$$

$$\deg v(\cdot) = 1, \text{ and} \tag{1.40}$$

$$\int_{\mathbb{R}^n} |\mathrm{D}v(x)|^n \,\mathrm{d}x \leqslant n^{n/2}\sigma_{n+1} + \eta, \tag{1.41}$$

for a given $\eta > 0$. For example, we can consider the function from [**21**, proof of Lemma 2] (see also [**39**, Lemma 3.2]), which, in our case, becomes

$$v(\varepsilon, x) = \begin{cases} \dfrac{1}{\varepsilon^4 + |x|^2}(2\varepsilon^2 \overline{x}, |x|^2 - \varepsilon^4), & |x| \leqslant \varepsilon \\ \left(\dfrac{\overline{x}}{|x|} A(|x|), \sqrt{1 - A^2(|x|)}\right), & |x| \in [\varepsilon, 2\varepsilon]\,, \\ (0,1) \in \mathbb{R}^n \times \mathbb{R}, & |x| \geqslant 2\varepsilon \end{cases}$$

where $\overline{x} \coloneqq ((-1)^{n+1}x_1, x_2, \dots, x_n)$ is used to control the sign of $\deg v(\varepsilon, \cdot)$, the map $A(\cdot)$ is an affine map chosen such that $v(\varepsilon, \cdot)$ is continuous, that is,

$$A(r) \coloneqq -\frac{2}{\varepsilon^2 + 1} r + \frac{4\varepsilon}{\varepsilon^2 + 1}, \quad \text{for } r > 0,$$

and $\varepsilon > 0$ is a small parameter that we will conveniently fix later.

The meaning of $\deg v(\varepsilon, \cdot)$ is that of the degree of $v(\varepsilon, \cdot)$ which is viewed as a map from the sphere $\mathbb{S}^n$ to itself, obtained by identifying $\mathbb{R}^n \cup \{\infty\}$ and $\mathbb{S}^n$ through the use of the stereographic projection π. So $\deg v(\varepsilon, \cdot)$ is given by

$$\begin{aligned} \deg v(\varepsilon, \pi(\cdot)) &= \frac{1}{\sigma_{n+1}} \int_{\mathbb{S}^n} \det\left[v(\varepsilon, \pi(y)), \partial_{y_1} v(\varepsilon, \pi(y)), \dots, \partial_{y_n} v(\varepsilon, \pi(y))\right] \mathrm{d}\mathcal{H}^n(y) \\ &= \frac{1}{\sigma_{n+1}} \int_{\mathbb{R}^n} \det\left[v(\varepsilon, x), \partial_{x_1} v(\varepsilon, x), \dots, \partial_{x_n} v(\varepsilon, x)\right] \mathrm{d}x, \end{aligned} \tag{1.42}$$

where $y_1 \dots, y_n$ are the coordinates of the point $y \in \mathbb{S}^n$ in an orthonormal basis of $\mathrm{T}_y\mathbb{S}^n$.

The function $v(\varepsilon, \cdot)$ clearly satisfies (1.39), and we next check that, provided ε is sufficiently small, properties (1.40) and (1.41) are also verified. Since

$$\det(v, \partial_{x_1} v, \dots, \partial_{x_n} v)(\varepsilon, x) = \begin{cases} \left(\dfrac{2\varepsilon^2}{\varepsilon^4 + |x|^2}\right)^n, & |x| \leqslant \varepsilon \\ \dfrac{2}{\varepsilon^2 + 1} \times \dfrac{A^{n-1}(|x|)}{|x|^{n-1}\sqrt{1 - A^2(|x|)}}, & |x| \in [\varepsilon, 2\varepsilon], \end{cases}$$

we have, after the appropriate change of variables, that

$$\deg v(\varepsilon, \cdot) = \frac{\sigma_n}{\sigma_{n+1}} \left(2^n \int_0^{1/\varepsilon} \frac{r^{n-1}}{(1 + r^2)^n} \,\mathrm{d}r + \int_0^{2\varepsilon/(\varepsilon^2+1)} \frac{r^{n-1}}{\sqrt{1 - r^2}} \,\mathrm{d}r\right). \tag{1.43}$$

We notice that the derivative with respect to ε of $\deg v(\varepsilon, \cdot)$ vanishes on the interval $(0, 1)$. Since the degree depends continuously on ε, we find, after another change of variables, that

$$\deg v(\varepsilon, \cdot) = \deg v(1, \cdot) = \frac{\sigma_n}{2\sigma_{n+1}} \left(2^n \int_0^1 \frac{t^{n/2-1}}{(1 + t)^n} \,\mathrm{d}t + \int_0^1 \frac{t^{n/2-1}}{(1 - t)^{1/2}} \,\mathrm{d}t\right). \tag{1.44}$$

The second integral is an Euler integral, and its value is

$$\int_0^1 t^{n/2-1}(1-t)^{1/2-1}\,\mathrm{d}t = \mathrm{B}\left(\frac{n}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}$$

where $\mathrm{B}(\cdot,\cdot)$ and $\Gamma(\cdot)$ are the classical Beta and Gamma functions, respectively. For the first integral in (1.44), we consider the hypergeometric function in its Euler integral representation, that is,

$$\mathrm{F}(\alpha,\beta;\gamma;\xi) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-\xi t)^{-\alpha}\,\mathrm{d}t$$

(see for example [**6**, Theorem 2.2.1]; here $\alpha,\beta,\gamma > 0$ and $\xi \in \mathbb{R}\setminus\{0\}$), and then use Kummer's formula

$$\mathrm{F}(\alpha,\beta;\alpha-\beta+1;-1) = \frac{\Gamma(\alpha-\beta+1)\Gamma\left(\frac{\alpha}{2}+1\right)}{\Gamma(\alpha+1)\Gamma\left(\frac{\alpha}{2}-\beta+1\right)}$$

(see [**6**, Corollary 3.1.2]). This gives us

$$\int_0^1 t^{n/2-1}(1+t)^{-n}\,\mathrm{d}t = \frac{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma(n+1)},$$

and, using the property $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$ satisfied by the Gamma function, we conclude that (1.40) is verified, that is,

$$\deg v(\varepsilon,\cdot) = 1, \quad \text{for any } \varepsilon \in (0,1). \tag{1.45}$$

In order to verify (1.41), we compute the differential of $v(\varepsilon,\cdot)$. We have that

$$|\mathrm{D}v(\varepsilon,x)| = \begin{cases} n^{1/2}\dfrac{2\varepsilon^2}{\varepsilon^4+|x|^2}, & |x| \leqslant \varepsilon \\ \left(\dfrac{4}{(\varepsilon^2+1)^2}\times\dfrac{1}{1-A^2(|x|)}+\dfrac{(n-1)A^2(|x|)}{|x|^2}\right)^{1/2}, & |x| \in [\varepsilon, 2\varepsilon]\,. \end{cases}$$

Thus,

$$\int_{|x|\leqslant\varepsilon} |\mathrm{D}v(\varepsilon,x)|^n\,\mathrm{d}x = n^{n/2}\sigma_n\int_0^\varepsilon\left(\frac{2\varepsilon^2}{\varepsilon^4+r^2}\right)^n r^{n-1}\,\mathrm{d}r = n^{n/2}\sigma_n 2^n\times I(\varepsilon),$$

where $I(\varepsilon)$ is the first integral in (1.43). From the fact that $\deg v(\varepsilon,\cdot) = 1$, and recalling (1.43), we can deduce that

$$I(\varepsilon) = \frac{\sigma_{n+1}}{2^n\sigma_n} - \frac{1}{2^n}\int_0^{2\varepsilon/(\varepsilon^2+1)}\frac{r^{n-1}}{\sqrt{1-r^2}}\,\mathrm{d}r,$$

for every $\varepsilon \in (0,1)$, and so

$$\lim_{\varepsilon\to 0}\int_{|x|\leqslant\varepsilon}|\mathrm{D}v(\varepsilon,x)|^n\,\mathrm{d}x = n^{n/2}\sigma_{n+1}.$$

On the other hand,

$$\begin{aligned}
&\int_{\varepsilon\leqslant|x|\leqslant 2\varepsilon} |\mathrm{D}v(\varepsilon,x)|^n \,\mathrm{d}x \\
&\qquad = 2^n\sigma_n \int_{\varepsilon}^{2\varepsilon} \left(\frac{1}{(\varepsilon^2+1)^2-4(2\varepsilon-r)^2} + (n-1)\frac{(2\varepsilon-r)^2}{r^2(\varepsilon^2+1)}\right)^{n/2} r^{n-1}\,\mathrm{d}r \\
&\qquad \leqslant 2^{2n}\sigma_n\varepsilon^n \left(\frac{1}{(1-\varepsilon^2)^2} + \frac{n-1}{\varepsilon^2+1}\right)^{n/2}
\end{aligned}$$

which tends to 0 when $\varepsilon \to 0$. So, we can choose $\varepsilon > 0$ sufficiently small for (1.41) to be verified, and for this function $v(\varepsilon,\cdot)$ we will just use the notation $v(\cdot)$.

The second step is to define, for every $k \geqslant 1$, the function $u : \Omega \smallsetminus \partial E \to \mathbb{S}^n$ by

$$u(\tilde{x},x) = \begin{cases} v\left(\dfrac{kx}{r-|\tilde{x}|}\right), & (\tilde{x},x) \in (\mathrm{pr}_{\mathbb{R}^m}E \times \mathbb{R}^n)\cap\Omega \\ (0,1) \in \mathbb{R}^n\times\mathbb{R}, & \text{elsewhere} . \end{cases}$$

We compute the differential of u to get

$$|\mathrm{D}u(\tilde{x},x)| = \left(\frac{k}{r-|\tilde{x}|}\right)\left[\left(\frac{|x|}{r-|\tilde{x}|}\right)^2+1\right]^{1/2} \times \left|\mathrm{D}v\left(\frac{kx}{r-|\tilde{x}|}\right)\right|,$$

if $(\tilde{x},x) \in \mathrm{pr}_{\mathbb{R}^m}E\times\mathbb{R}^n$. For k sufficiently large, and η conveniently chosen, we have

$$\begin{aligned}
&\int_\Omega a(x)|\mathrm{D}u(x)|^n\,\mathrm{d}x \\
&\qquad = \int_{\mathrm{pr}_{\mathbb{R}^m}E\cap\Omega}\left(\int_{\mathbb{R}^n\cap\Omega} a\left(\tilde{x},\frac{y(r-|\tilde{x}|)}{k}\right)\left[\left(\frac{|y|}{k}\right)^2+1\right]^{n/2}\times|\mathrm{D}v(y)|^n\,\mathrm{d}y\right)\mathrm{d}\tilde{x} \\
&\qquad \leqslant n^{n/2}\sigma_{n+1}\int_{\mathrm{pr}_{\mathbb{R}^m}E\cap\Omega} a(\tilde{x},0)\,\mathrm{d}\tilde{x} + \delta = n^{n/2}\sigma_{n+1}\mathcal{H}^m_a(E)+\delta,
\end{aligned}$$

so inequality (1.38) is satisfied.

Since the map u belongs to $\mathrm{W}^{1,n}_{\mathrm{loc}}(\Omega,\mathbb{S}^n)$, and is locally Lipschitz outside ∂E, we know that

$$\star\mathrm{J}u = \frac{\sigma_{n+1}}{n+1}\deg(u,\partial E)[\![\partial E]\!],$$

where the degree of u along the curve ∂E means the degree of the restriction of u to the boundary of any disk $D\subset\mathbb{R}^{m+n}$ of dimension $(n+1)$, that intersects transversally ∂E in only one point, and whose orientation τ_D is such that

$$\tau_{\partial E}\wedge\tau_D = e_1^{(m+n)}\wedge\cdots\wedge e_{m+n}^{(m+n)},$$

$\tau_{\partial E}$ being the orientation of the boundary ∂E induced by that of E. If we choose

$$D = \left\{\xi = \tilde{x}_1e_1^{(m+n)} + x_1e_{m+1}^{(m+n)}+\cdots+x_ne_{m+n}^{(m+n)} \text{ with } |\xi - re_1^{(m+n)}|\leqslant r\right\},$$

then it can be easily seen that $\deg(u,\partial D) = \deg v(\cdot) = 1$, and, consequently, the map u belongs to the class $\mathcal{E}([\![\partial E]\!])$. □

Remark 1.6. — *We could also compute the degree of the map $v(\varepsilon,\cdot)\circ\pi\colon\mathbb{S}^n\to\mathbb{S}^n$ by using, instead of* (1.42), *the formula*

$$\deg v(\varepsilon,\pi(\cdot)) = \sum_{x\in(v\circ\pi)^{-1}(y)} \det \mathrm{J}(v\circ\pi)(x), \tag{1.46}$$

where $y\in\mathbb{S}^n$ is any regular value of $v\circ\pi$. Remark that, since $v\circ\pi$ is Lipschitz, (1.46) *makes sense if the sum is taken over the points x where the function is approximately differentiable and this definition coincides with the classical definition of the degree (*[**30**, Sections 3.2, 3.3]*).*

For $y\coloneqq(0,-1)\in\mathbb{R}^n\times\mathbb{R}$, we have $(v\circ\pi)^{-1}(y)=\{y\}$ and

$$\det \mathrm{J}(v\circ\pi)(y) = \frac{1}{\varepsilon^{2n}},$$

which is strictly positive, and hence, using (1.42), *we find* (1.45)*.*

Yet another way of obtaining (1.45) *could be to find a homotopy between $v\circ(\varepsilon,\cdot)$ and the identity of $\mathbb{S}^n$.*

For passing from the boundary of a disk to the general case of the boundary of a rectifiable current, we will need the following result, which is a variant of [**3**, Corollary 7.13].

Theorem 1.7. — *Let T be a current in $\mathcal{R}_m(\Omega)$ with finite mass, and suppose that $a(\cdot)\colon\Omega\to(0,\infty)$ is a bounded continuous function. Given $\rho(\cdot)$ a strictly positive function on Ω, and $s>1$, there exist, for any $\varepsilon>0$, a rectifiable current $R\in\mathcal{R}_m(\Omega)$, and finitely many m-dimensional disks $E_{r_i}(x_i)\subset\Omega$ of radii $r_i\leqslant\rho(x_i)$, which satisfy the following properties:*

(a) $\partial T=\sum_i \partial[\![E_{r_i}(x_i)]\!]+\partial R$;

(b) $\sum_i \mathcal{H}^m_a(E_{r_i}(x_i))\leqslant \dfrac{1+\varepsilon}{s^m}\,\mathbb{M}(T\llcorner a)$;

(c) $\mathbb{M}(R\llcorner a)\leqslant\left(1-\dfrac{1}{s^{m+1}}\right)\mathbb{M}(T\llcorner a)$;

(d) *the balls $\overline{B}_{sr_i}(x_i)$ are pairwise disjoint and contained in Ω.*

Proof. — This result is a direct consequence of [**3**, Theorem 7.12] (which, in turn, is proved using the approximation of integral flat currents by integral polyhedral currents – [**26**, Thm. 4.2.22]).

We begin, just like in the proof of proposition 1.3, by applying the Vitali-Besicovitch covering theorem. Since $a(\cdot)$ is uniformly continuous on the compact support of T, then, for any $\eta>0$, we can find $\gamma>0$ such that, $|a(x)-a(y)|\leqslant\eta$ whenever $|x-y|\leqslant\gamma$, and, considering T given by the integral representation (1.25), we cover $\widetilde{T}$, up to a set of small $\mathcal{H}^m$-measure, by a finite and disjoint family of small balls

$$\mathcal{H}^m\left(\widetilde{T}\setminus\bigcup_{j=1}^{p}B_{\gamma_j}(\xi_j)\right)\leqslant\lambda\,,\quad\text{where}\quad \lambda>0,\ \gamma_j\leqslant\gamma,\ \xi_j\in\widetilde{T}.$$

The theorem mentioned above states that the currents $T\llcorner B_{\gamma_j}(\xi_j)$ can be approximated in the flat norm by finite sums $\sum_{i=1}^{p_j}[\![E_{\rho_{ij}}(x_{ij})]\!]$ of m-dimensional disks with radii $\rho_{ij}<\rho(x_{ij})$ and such that $\overline{B}_{\rho_{ij}}(x_{ij})$ are pairwise disjoint and contained in $B_{\gamma_j}(x_j)$.

This implies that, for any $\delta > 0$, there exist rectifiable currents $R_j \in \mathcal{R}_m(B_{\gamma_j}(x_j))$ and $Q_j \in \mathcal{R}_{m+1}(B_{\gamma_j}(x_j))$, that satisfy

$$T \llcorner B_{\gamma_j}(\xi_j) = \sum_{i=1}^{p_j} [\![E_{\rho_{ij}}(x_{ij})]\!] + R_j + \partial Q_j \tag{1.47}$$

$$\mathbb{M}(R_j) \leqslant \delta\, \mathbb{M}\left(T \llcorner B_{\gamma_j}(\xi_j)\right), \quad \text{and} \tag{1.48}$$

$$\sum_{i=1}^{p_j} \mathcal{H}^m(E_{\rho_{ij}}(x_{ij})) \leqslant (1+\delta)\, \mathbb{M}\left(T \llcorner B_{\gamma_j}(\xi_j)\right), \ \forall j = 1\ldots, p. \tag{1.49}$$

We rescale the radii by taking $r_{ij} := \rho_{ij}/s$. Define $R_0 := T \llcorner A$, where the measurable set A represents the difference $\widetilde{T} \setminus \bigcup_{j=1}^{p} B_{\gamma_j}(\xi_j)$, and consider the following m-dimensional rectifiable current

$$R := \sum_{j=1}^{p}\left[\sum_{i=1}^{p_j}\left([\![E_{\rho_{ij}}(x_{ij})]\!] - [\![E_{r_{ij}}(x_{ij})]\!]\right) + R_j\right] + R_0.$$

After summing the identities in (1.47), we obtain

$$T = \sum_{j=1}^{p}\sum_{i=1}^{p_j} [\![E_{ij}(x_{ij})]\!] + R + \sum_j \partial Q_j,$$

so we need only to apply the boundary operator, and reindex the finite number of disks $E_{r_{ij}}(x_{ij})$, to get the first property required.

To arrive at the second one, we approximate $a(x)$ with $a(\xi_j)$ on each $B_{\gamma_j}(\xi_j)$, and use the inequalities in (1.49), that give us

$$\begin{aligned}\sum_{j=1}^{p}\sum_{i=1}^{p_j} \mathcal{H}^m_a(E_{r_{ij}}(x_{ij})) &\leqslant \sum_{j=1}^{p}\sum_{i=1}^{p_j}(a(\xi_j)+\eta)\mathcal{H}^m(E_{r_{ij}}(x_{ij}))\\ &\leqslant \frac{1+\delta}{s^m}(\mathbb{M}(T\llcorner a) + 2\eta\mathbb{M}(T)).\end{aligned}$$

Then, using also the inequalities (1.48), and the fact that A has small $\mathcal{H}^m$ measure, we see that

$$\begin{aligned}\mathbb{M}(R\llcorner a) &= \sum_{j=1}^{p}\sum_{i=1}^{p_j}\left[\mathbb{M}([\![E_{\rho_{ij}}(x_{ij})]\!]\llcorner a) - \mathbb{M}([\![E_{r_{ij}}(x_{ij})]\!]\llcorner a)\right]\\ &\quad + \sum_{j=1}^{p}\mathbb{M}(R_j\llcorner a) + \mathbb{M}(R_0\llcorner a)\\ &\leqslant \left[\left(1-\frac{1}{s^m}\right)(1+\delta)+\delta\right](\mathbb{M}(T\llcorner a) + 2\eta\mathbb{M}(T)) + \lambda_0,\end{aligned}$$

where λ_0 approaches 0, as λ tends to 0. With a proper choice of small η and δ, these two last inequalities yield the properties b. and c. in the statement of the theorem. □

1.4.3.1. Proof of the upper bound. — Making use of the two results from above, we will see that the exact strategy from [**3**, Theorem 5.6] works effectively for proving the precise upper bound of the weighted energy. Let M be a current in $\mathcal{C}(\Gamma)$ and let $\delta > 0$, and $s \in (1,2)$. The result is obtained by finding sequences of maps $u_j \in \mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$, closed subsets $S_j \subset \Omega$ of dimension $m-1$ such that $u_j \in \mathrm{Lip}_{\mathrm{loc}}(\Omega \setminus S_j)$, and currents $R_j \in \mathcal{R}_m(\Omega)$ with the following properties:

$$\star \mathrm{J}u_j = \frac{\sigma_{n+1}}{n+1}(\Gamma - \partial R_j) \tag{1.50}$$

$$\mathbb{M}(R_j \llcorner a) \leqslant c_1\, \mathbb{M}(R_{j-1} \llcorner a), \text{ with } c_1 < 1 \tag{1.51}$$

$$\int_\Omega a(x)\, |\mathrm{D}u_j(x) - \mathrm{D}u_{j-1}(x)|^n \, \mathrm{d}x \leqslant c_2^n\, \mathbb{M}(R_{j-1} \llcorner a), \tag{1.52}$$

with c_1 and c_2 constants verifying $\left(\dfrac{c_2}{1-c_1^{1/n}}\right)^n = n^{n/2}\sigma_{n+1} + \dfrac{\delta}{\mathbb{M}(M \llcorner a)}$. These sequences will be constructed by induction, starting from u_0 constant, $S_0 = \varnothing$ and $R_0 = M$.

Let's see first how these sequences help us prove the upper bound. From (1.51) and (1.52) we deduce that $(\mathrm{D}u_j)_j$ (modulo a subsequence) converges in $\mathrm{L}^n(\Omega, \mathbb{R}^{n+1})$, so we can assume that $(u_j)_j$ converges to some map u in $\mathrm{W}^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{S}^n)$. Hence, $\star\mathrm{J}u_j \rightharpoonup \star\mathrm{J}u$, so, by (1.50) and (1.52), we get $\star\mathrm{J}u = \dfrac{\sigma_{n+1}}{n+1}\Gamma$ and, by (1.51) and (1.52) and the condition on the constants c_1 and c_2, we have

$$\int_\Omega a(x)|\mathrm{D}u(x)|^n \, \mathrm{d}x \leqslant n^{n/2}\sigma_{n+1}\mathbb{M}(M \llcorner a) + \delta.$$

For the construction of the sequences, suppose we have u_{j-1}, S_{j-1} and R_{j-1} with the desired properties, for a fixed j. Since S_{j-1} is closed, for each $x \in \Omega \setminus S_{j-1}$ we can find $r(x) > 0$ such that $B_{r(x)}(x) \subset \Omega \setminus S_{j-1}$. The function u_{j-1} is locally Lipschitz outside S_{j-1}, so we have $\|Du_{j-1}\|_{\mathrm{L}^\infty(B_{r(x)}(x))} < \infty$ and we can define

$$0 < \rho(x) := \min\left\{\frac{r(x)}{s}\ ;\ \frac{s-1}{2s}\|Du_{j-1}\|^{-1}_{\mathrm{L}^\infty(B_{r(x)}(x))}\right\}, \tag{1.53}$$

for every x in $\Omega \setminus S_{j-1}$. Then, by Theorem 1.7, applied with $T = R_{j-1}$, for every small $\varepsilon > 0$ we get finitely many m-dimensional disks $E_{r_i}(x_i)$ in $\mathbb{R}^{m+n}$ with $r_i < \rho(x_i)$ and a rectifiable m-current $R_j \subset \Omega$ such that the balls $\overline{B}_{sr_i}(x_i)$ are pairwise disjoint, and

(a) $\partial R_{j-1} = \sum_i \partial [\![E_{r_i}(x_i)]\!] + \partial R_j$

(b) $\sum_i \mathcal{H}^m_a(E_{r_i}(x_i)) \leqslant \dfrac{1+\varepsilon}{s^m}\mathbb{M}(R_{j-1} \llcorner a)$

(c) $\mathbb{M}(R_j \llcorner a) \leqslant \left(1 - \dfrac{1}{s^{m+1}}\right)\mathbb{M}(R_{j-1} \llcorner a)$.

On every $E_{r_i}(x_i)$ we insert the dipole given by Lemma 1.5. More precisely, for any $\eta > 0$, we take some maps $v_i \in \mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$, which are locally Lipschitz on $\Omega \setminus \partial E_{r_i}(x_i)$, constantly equal to some $y_0 \in u_{j-1}(B_{sr_i}(x_i) \setminus \overline{B}_{r_i}(x_i))$, and that satisfy

$$\int_\Omega a(x)\, |\mathrm{D}v_i|^n(x)\, \mathrm{d}x \leqslant \sigma_{n+1} n^{n/2}\mathcal{H}^m_a(E_{r_i}(x_i)) + \eta$$

and thus,

$$\star\mathrm{J}v_i = \frac{\sigma_{n+1}}{n+1}\,\partial E_{r_i}(x_i).$$

Because v_i agrees with one of the values of u_{j-1} on the annulus $B_{sr_i}(x_i) \setminus \overline{B}_{r_i}(x_i)$, for each i, we have

$$\begin{aligned}
&\|u_{j-1}-v_i\|_{\mathrm{L}^\infty(B_{sr_i}(x_i)\setminus\overline{B}_{r_i}(x_i))} \\
&\qquad \leqslant \sup_{x,\xi\in B_{sr_i}(x_i)} |u_{j-1}(x)-u_{j-1}(\xi)| = \sup_{\substack{h\in B_{2sr_i}\\ \xi\in B_{sr_i}(x_i)}} |u_{j-1}(\xi+h)-u_{j-1}(\xi)| \\
&\qquad \leqslant \sup_{h\in B_{2sr_i}} \|\mathrm{D}u_{j-1}\|_{\mathrm{L}^\infty(B_{sr_i}(x_i))}|h| = 2sr_i\|\mathrm{D}u_{j-1}\|_{\mathrm{L}^\infty(B_{sr_i}(x_i))}.
\end{aligned}$$

By the choice of ρ, this implies $|u_{j-1}-v_i| \leqslant s-1 \leqslant 1$ a.e. on $B_{sr_i}(x_i) \setminus \overline{B}_{r_i}(x_i)$, and the balls $B_{sr_i}(x_i)$ being pairwise disjoint, we can repeatedly apply Lemma 5.4 in [**3**] to get the maps $w_i \in \mathrm{W}^{1,n}(\Omega,\mathbb{S}^n)$ with

$$w_i = \begin{cases} v_i, & \text{on } B_{r_i}(x_i) \\ w_{i-1}, & \text{on } \Omega\setminus B_{sr_i}(x_i) \end{cases}, \qquad \text{where } w_0 \equiv u_{j-1}.$$

Since there are a finite number of balls $B_{r_i}(x_i)$, we can take u_j to be the last w_i, that is,

$$u_j = \begin{cases} v_i, & \text{on } B_{r_i}(x_i),\ \forall i \\ u_{j-1}, & \text{on } \Omega\setminus \bigcup_i B_{sr_i}(x_i) \end{cases};$$

moreover, Lemma 5.4 in [**3**] also gives us $\mathrm{J}u_j = \mathrm{J}u_{j-1} + \sum_i \mathrm{J}v_i$, and

$$\|\mathrm{D}u_j\|_{\mathrm{L}^\infty(B_{sr_i}(x_i)\setminus\overline{B}_{r_i}(x_i))} \leqslant \frac{2}{\sqrt{3}}\frac{3s-1}{s-1}\|\mathrm{D}u_{j-1}\|_{\mathrm{L}^\infty(B_{sr_i}(x_i))}, \quad \forall i.$$

By construction (see the proof of Lemma 5.4 in [**3**] and recall that $r_i < \rho(x_i)$ with $\rho(\cdot)$ given by (1.53)), the function u_j is locally Lipschitz outside $S_j \coloneqq \bigcup_i \partial E_{r_i}(x_i) \cup S_{j-1}$. Properties (a) and (c) show that u_j satisfies the required conditions (1.50), and (1.51) with $c_1 \coloneqq 1-\dfrac{1}{s^{m+1}}$, respectively, and it remains to check (1.52). We have the estimate

$$\begin{aligned}
&\left(\int_\Omega a(x)|\mathrm{D}u_j(x)-\mathrm{D}u_{j-1}(x)|^n\,\mathrm{d}x\right)^{1/n} \\
&\quad \leqslant \left(\sum_i \int_{B_{sr_i}(x_i)\setminus\overline{B}_{r_i}(x_i)} a(x)|\mathrm{D}u_j|^n\,\mathrm{d}x + \sum_i \int_{B_{r_i}(x_i)} a(x)|\mathrm{D}v_i|^n\,\mathrm{d}x\right)^{1/n} \\
&\quad + \left(\sum_i \int_{B_{sr_i}(x_i)} a(x)|\mathrm{D}u_{j-1}|^n\,\mathrm{d}x\right)^{1/n} \\
&\quad \leqslant \left[\left(\frac{3s-1}{\sqrt{3}s}\right)^n \sum_i \frac{1}{r_i^n}\int_{B_{sr_i}(x_i)\setminus\overline{B}_{r_i}(x_i)} a(x)\,\mathrm{d}x + n^{n/2}\sigma_{n+1}\sum_i \mathcal{H}_a^m(E_{r_i}(x_i)) + \sum_i \eta\right]^{1/n} \\
&\quad + \frac{s-1}{2s}\left(\sum_i \frac{1}{r_i^n}\int_{B_{r_i}(x_i)} a(x)\,\mathrm{d}x\right)^{1/n}.
\end{aligned}$$

Taking into account property (b), it is clear that there exist $\varepsilon > 0$, $\eta > 0$ and $s \in (1,2)$, such that condition (1.52) is verified, with the constant c_2 also satisfying the required inequality relative to c_1.

1.4.3.2. Proof of the lower bound. — As we mentioned in the introduction, the lower bound can be obtained with the help of the coarea formula, like in [**3**], and, before that, as seen in [**4**]. Let $u : \Omega \to \mathbb{S}^n$ be an element of $\mathcal{E}(\Gamma)$, that is, u belongs to $\mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$ and it satisfies $\star \mathrm{J}u = \dfrac{\sigma_{n+1}}{n+1}\Gamma$. We can choose, recalling (1.36), a point $z \in \mathbb{S}^n$ such that $\star \mathrm{J}u = \dfrac{\sigma_{n+1}}{n+1}[\![N_z]\!]$, and which verifies the inequality

$$\int_{N_z} a(x)\,\mathrm{d}\mathcal{H}^m(x) \leqslant \frac{1}{\sigma_{n+1}} \int_{\mathbb{S}^n} \left(\int_{N_y} a(x)\,\mathrm{d}\mathcal{H}^m(x) \right) \mathrm{d}\mathcal{H}^n(y).$$

By the coarea formula (1.32), and inequality (1.27), we have

$$\int_{\mathbb{S}^n} \left(\int_{N_y} a(x)\,\mathrm{d}\mathcal{H}^m(x) \right) \mathrm{d}\mathcal{H}^n(y) = \int_{\Omega} a(x) |u^\sharp \omega_0(x)|\,\mathrm{d}x \leqslant n^{-n/2} \int_{\Omega} a(x) |\mathrm{D}u(x)|^n\,\mathrm{d}x.$$

Since the current $M_0 := (-1)^m [\![N_z]\!]$ belongs to the class $\mathcal{C}(\Gamma)$, we have

$$\int_{N_z} a(x)\,\mathrm{d}\mathcal{H}^m(x) = \mathbb{M}(M_0 \llcorner a) = \mathbb{M}([\![N_z \llcorner a]\!]) \geqslant \inf_{M \in \mathcal{C}(\Gamma)} \mathbb{M}(M \llcorner a),$$

and therefore we have obtained

$$\int_{\Omega} a(x) |\mathrm{D}u(x)|^n\,\mathrm{d}x \geqslant n^{n/2} \sigma_{n+1} \inf_{M \in \mathcal{C}(\Gamma)} \mathbb{M}(M \llcorner a),$$

which completes the proof of the theorem.

1.5. Preuve du Théorème ($\star$)

Cette section contient la preuve du théorème ($\star$) de la section 1.2. Nous reprenons ici l'énoncé:

Theorem ($\star$). — *Let Ω be a smooth, bounded and connected open set in $\mathbb{R}^{m+n}$, and let $u \in \mathrm{W}^{1,n}(\Omega; \mathbb{S}^n)$. Then, u can be approximated in the $\mathrm{W}^{1,n}$ norm by maps u_k belonging to $\mathrm{W}^{1,n}(\Omega; \mathbb{S}^n) \cap \mathrm{C}^\infty(\Omega \setminus \Gamma_k; \mathbb{S}^n)$, where Γ_k is the boundary of a surface in $\mathbb{R}^{n+1}$.*

Moreover, for almost every $y \in \mathbb{S}^n$, we have $\partial(u_k^{-1}(y)) = \Gamma_k$.

The proof follows closely the proof in [**14**] of the density of the class $\mathscr{R}$ in $\mathrm{W}^{s,p}(\mathbb{S}^N, \mathbb{S}^1)$.

Proof. — We start by approximating u with smooth functions v_k (extend u to the whole space and then convolute with a regularizing sequence) and such that $v_k \to u$ and $\mathrm{D}v_k \to \mathrm{D}u$ almost everywhere and dominated.

For fixed points $a \in \mathbb{R}^{n+1}$, consider the projections

$$\pi_a : \mathbb{R}^{n+1} \to \mathbb{S}^n, \qquad \pi_a(x) \coloneqq \frac{x-a}{|x-a|},$$

and define

$$v_k^a \coloneqq \pi_a \circ v_k.$$

The maps v_k^a are smooth outside $v_k^{-1}(a)$, which, by Sard's Theorem, for almost all a, is a smooth manifold of dimension $m-1$ in $\mathbb{R}^{m+n}$. For $|a| \leqslant 1/2$, we denote by j_a the smooth inverse of the restriction of π_a to $\mathbb{S}^n$, and we define the maps

$$w_k^a \coloneqq j_a \circ v_k^a.$$

Then w_k^a belongs to $\mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$ and are smooth outside $v_k^{-1}(a)$. Also, we show that w_k^a converges to u in $\mathrm{W}^{1,n}$ for almost all $a \in B_{1/2}^{n+1}$. Indeed, as u takes values in $\mathbb{S}^n$, we can write $u = j_a \circ \pi_a \circ u$ and then, by dominated convergence, we find that $w_k^a \to u$ in L^n. On the other hand, we have

$$\begin{aligned}|\mathrm{D}w_k^a(x) - \mathrm{D}u(x)| &\leqslant |\mathrm{D}(j_a \circ \pi_a)(u_k(x)) - \mathrm{D}(j_a \circ \pi_a)(u(x))| \cdot |\mathrm{D}u_k(x)| \\ &\quad + |\mathrm{D}(j_a \circ \pi_a)(u(x))| \cdot |\mathrm{D}u_k(x) - \mathrm{D}u(x)| =: I_k(x,a) + J_k(x,a).\end{aligned}$$

We note that $|\mathrm{D}(j_a \circ \pi_a)(y)| \lesssim \dfrac{1}{|y-a|}$, therefore

$$\int_\Omega J_k(x,a)^n \,\mathrm{d}x \lesssim \int_\Omega |\mathrm{D}v_k(x) - \mathrm{D}u(x)|^n \,\mathrm{d}x \longrightarrow 0$$

for all a with $|a| \leqslant \dfrac{1}{2}$. We now turn to the term $I_k(x,a)$. Since

$$\mathcal{H}^{m+n}\left\{x \in \Omega \;\middle|\; |v_k(x)| \leqslant \frac{3}{4}\right\} \longrightarrow 0 \quad \text{when } k \to \infty$$

and the integral $\displaystyle\int_{B_{1/2}^{n+1}} |y-a|^{-n} \,\mathrm{d}a$ converges and can be majorated by a constant independent of y for any $y \in B_1^{n+1}$, we have

$$\int_{B_{1/2}^{n+1}} \int_{[|v_k(x)| \leqslant 3/4]} I_k(x,a) \;\; dx\,\mathrm{d}a \longrightarrow 0. \tag{1.54}$$

On the other hand, on $B_1^{n+1} \setminus B_{3/4}^{n+1}$, the map $j_a \circ \pi_a$ is smooth, and hence, by dominated convergence, we also have

$$(1.55) \quad \int_{B_{1/2}^{n+1}} \int_{[|v_k(x)| \geqslant 3/4]} I_k(x,a)\, \mathrm{d}x\, \mathrm{d}a \longrightarrow 0,$$

By combining (1.54) and (1.55), we obtain that $w_k^a \to u$ in $\mathrm{W}^{1,n}(\Omega, \mathbb{S}^n)$, for almost all $a \in B_{1/2}^{n+1}$.

The preimage of a point $y \in \mathbb{S}^n$ under the map w_k^a is the set $v_k^{-1}(D_a(y))$, where $D_a(y)$ is the ray with origin a which passes through y. It is a smooth m-dimensional manifold with boundary and $\partial(w_k^a)^{-1}(y) = v_k^{-1}(a)$. Indeed, if a is such that v_k is transversal to $D_a(y) \setminus \{a\}$, then, by the Preimage theorem for manifolds ([**32**, p. 28]), $v_k^{-1}(D_a(y) \setminus \{a\})$ is a (boundaryless) manifold of dimension m. Also, if a is a regular value of v_k, then, for each point $x \in v_k^{-1}(a)$ we have that $\operatorname{rank} \mathrm{d}(v_k)_a = n+1$, so locally the map v_k is equivalent to the projection on $\mathbb{R}^{n+1}$ and therefore $v_k^{-1}(D_a(y))$ becomes a manifold with boundary and its boundary is $v_k^{-1}(a)$. Since the set of such points a is dense in $\mathbb{R}^{n+1}$, by Sard's Theorem and a consequence to the Transversality theorem for families of mappings ([**32**, p. 69]), the conclusion is obtained with $u_k = w_k^a$, for a conveniently chosen. □

CHAPITRE 2 RELÈVEMENTS DES APPLICATIONS À VALEURS DANS LE CERCLE

2.1. Introduction

Soit Ω un domaine de $\mathbb{R}^n$ régulier, borné et simplement connexe. Considérons les applications mesurables $u: \Omega \to \mathbb{S}^1$, c'est-à-dire $u(x) \in \mathbb{C}$ et $|u(x)| = 1$, pour presque tout $x \in \Omega$. Si u est de classe C^k (où $k \in \mathbb{N}$), le théorème classique du relèvement nous donne l'existence (et unicité modulo un multiple de 2π) d'un relèvement de classe C^k de u : une application $\varphi: \Omega \to \mathbb{R}$ qui vérifie

$$u(x) = \exp(\mathrm{i}\varphi(x)), \quad \forall\, x \in \Omega. \tag{2.1}$$

Nous disons alors que les espaces $X = \mathrm{C}^k(\Omega; \mathbb{S}^1)$ ont la propriété du relèvement : toute application $u \in X$ admet un relèvement φ de la même régularité que u. L'application φ s'appelle aussi *phase* ou *argument* de u.

La propriété du relèvement des espaces de Sobolev $\mathrm{W}^{s,p}(\Omega; \mathbb{S}^1)$ a été étudié en détail par Bourgain, Brezis et Mironescu dans [**10**]. Les auteurs ont donné une caractérisation complète des $\mathrm{W}^{s,p}$ qui satisfont cette propriété, selon les valeurs de s et p :

Théorème ([**10**]). — *Soit $1 \leqslant p < \infty$. Les espaces $\mathrm{W}^{s,p}(\Omega; \mathbb{S}^1)$ ont la propriété du relèvement si et seulement si s et p se trouvent dans l'une des situations ci-dessous :*

(i) $0 < s < 1$ *et* $sp < 1$ *;*

(ii) $0 < s < 1$ *et* $sp \geqslant n$ *;*

(iii) $1 \leqslant s < \infty$ *et* $sp \geqslant 2$.

La condition suffisante (iii) coïncide pour $s = 1$ avec un résultat de Bethuel et Zheng ([**8**, Lemma 1]).

Le cas (ii) est traité de manière suivante. Dans le cas où $sp > n$, la phase peut être définie ponctuellement :

$$\varphi(x) := -\mathrm{i} \ln u(x).$$

Cela est une conséquence du fait que, par le théorème du plongement de Sobolev, $\mathrm{W}^{s,p} \hookrightarrow \mathrm{C}^0$ si $sp > n$ et donc il existe une détermination continue de $\ln u$. Dans le cas où $sp = n$, après des extensions répétées de u en augmentant s à chaque fois, le problème se réduit finalement à une application de (iii).

Le cas le plus délicat est (i). La preuve est constructive : les auteurs proposent une construction itérative de la phase φ (avec un passage à la limite). Nous reviendrons sur cette construction dans la Section 2.2. Les auteurs soulèvent aussi la question du contrôle de $|\varphi|_{\mathrm{W}^{s,p}}$ par $|u|_{\mathrm{W}^{s,p}}$. L'estimation qui résulte directement de la construction montre une dépendance linéaire

$$(2.2) \qquad |\varphi|_{\mathrm{W}^{s,p}} \leqslant C(s,p,n)\, |u|_{\mathrm{W}^{s,p}}.$$

Cette relation n'existe pas dans le cas $\mathrm{H}^{1/2}$ ou, plus généralement, quand $sp = 1$. En effet, il y a des exemples de suites $(u_k)_k$ uniformément bornées dans $\mathrm{W}^{1/p,p}$ dont les phases (uniques) vérifient $|\varphi_k|_{\mathrm{W}^{1/p,p}} \to +\infty$.

Les semi-normes $|\cdot|_{\mathrm{W}^{s,p}}$ employées dans (2.2) et tout au long du chapitre représentent une semi-norme équivalente à la norme habituelle sur l'espace quotient $\mathrm{W}^{s,p}/\mathbb{C}$. En général, la semi-norme utilisée dans le cas $0 < s < 1$ est celle de Gagliardo :

$$|\varphi|_{\mathrm{W}^{s,p}} = \int_\Omega \int_\Omega \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{n+sp}} \mathrm{d}x\mathrm{d}y.$$

Remarquons qu'il n'y a pas de sens de considérer des vraies normes dans les estimations de la forme

$$(2.3) \qquad |\varphi|_{\mathrm{W}^{s,p}} \leqslant F\left(|u|_{\mathrm{W}^{s,p}}\right),$$

car si φ est une phase de u, alors $\varphi + 2\pi$ est aussi une phase de u, donc il faut que $|\varphi|_{\mathrm{W}^{s,p}}$ ne tienne pas compte des constantes additives.

La quantité $C(s,p,n)$ dans (2.2) est de la forme

$$(2.4) \qquad C(s,p,n) = c(p,n)\, \frac{1}{s(1-sp)}.$$

Se pose maintenant la question de l'optimalité de cette estimation : est-ce qu'il existe une autre dépendance de la forme (2.4) telle qu'elle soit vérifiée pour chaque $u \in \mathrm{W}^{s,p}$, avec $sp < 1$, et pour une des phases possibles de u ? Nous nous sommes intéressés principalement au comportement de $C(s,p,n)$ au voisinage du cas critique $sp = 1$. Le p et le n sont donc supposés fixés et nous suivons une éventuelle baisse de la puissance de $1 - sp$ dans (2.4).

Il s'avère qu'un contrôle meilleur que celui donné par (2.4) est possible, mais pour une autre phase de u que celle construite dans la preuve de (i). Il s'agit d'une estimation améliorée qui pour $p = 2$ est de la forme

$$(2.5) \qquad |\varphi|_{\mathrm{H}^s} \leqslant C\, \frac{1}{(1-2s)^{1/2}}\, |u|_{\mathrm{H}^s},$$

où $C = C(s,p,n)$ indépendant de $s \to 1/2$. Ce résultat à été démontré dans [**10**] à l'aide d'une méthode des moyennes que nous présentons aussi dans la section suivante.

Nous avons montré dans [**43**] que (2.5) s'étend au cas $1 \leqslant p < \infty$ quelconque :

$$(2.6) \qquad |\varphi|_{\mathrm{W}^{s,p}} \leqslant C(p,n)\, \frac{1}{s(1-sp)^{1/p}}\, |u|_{\mathrm{W}^{s,p}}.$$

La preuve de ce résultat reprend l'idée des moyennes. Par contre, comme (2.5) s'appuie sur des techniques d'analyse de Fourier, propres au cas L^2, il a été nécessaire de développer des méthodes alternatives pour (2.5).

Il reste ainsi d'établir l'optimalité de (2.5) et, en général de (2.6). L'optimalité des estimations de la forme (2.3) signifie qu'il existe une application $u \in \mathrm{W}^{s,p}$ et une phase φ de u qui vérifie (2.3) avec égalité à une constante multiplicative près.

Pour le cas $p = 2$, $n \geqslant 2$ cela a été démontré déjà dans [**10**], et dans le cas plus délicat $n = 1$ le résultat d'optimalité a été démontré par les mêmes auteurs dans un article ultérieur [**11**]. La preuve utilise l'analyse de Fourier sur L^1 (multiplicateurs de Fourier) et repose sur le comportement de la meilleure constante dans le plongement $\mathrm{W}_0^{1-\varepsilon}(]0,1[) \hookrightarrow \mathrm{L}^{1/\varepsilon}(]0,1[)$. Dans [**43**], nous avons démontré que l'optimalité de la constante dans (2.6) reste valable dans le cas plus général $p > 1$. Même si les arguments de ces deux résultats sont apparentés, la nouvelle preuve est plus simple que celle initiale pour $p = 2$.

Le seul cas dans lequel l'estimation (2.6) n'est pas optimale est $p = 1$. Dans ce cas-ci, il existe toujours un relèvement φ de $u \in \mathrm{W}^{s,1}(\mathbb{T}^n)$ tel que le quotient entre les semi-normes de φ est de u soit uniformément borné (par rapport à $0 < s < 1$). En effet, l'inégalité (2.6) peut être amélioré à

$$|\varphi|_{\mathrm{W}^{s,1}} \leqslant 2\,|u|_{\mathrm{W}^{s,1}} \tag{2.7}$$

([**43**, Proposition 1.6]). D'autre part, l'inégalité réciproque $|\varphi|_{\mathrm{W}^{s,p}} \geqslant |u|_{\mathrm{W}^{s,p}}$ est toujours vraie, car

$$|\exp(\mathrm{i}\varphi(x)) - \exp(\mathrm{i}\varphi(y))| = 2\left|\sin\left(\frac{\varphi(x)-\varphi(y)}{2}\right)\right| \leqslant 2\left|\frac{\varphi(x)-\varphi(y)}{2}\right| = |\varphi(x)-\varphi(y)|.$$

Donc (2.7) est clairement optimale (du fait que nous nous n'intéressons pas au constantes multiplicatives absolues). La preuve de ce résultat suit celle de l'existence des relèvements BV de Dávila et Ignat ([**23**]) et d'une simplification de la preuve de ce résultat par Merlet ([**38**]).

2.2. La construction d'une phase quand $sp < 1$ et la méthode des moyennes

Pour une simplification d'ordre technique, nous considérons $\Omega = (0,1)^n = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$ et les applications $u\colon \Omega \to \mathbb{S}^1$ périodiques. Les normes et les semi-normes d'une telle application sont donc prise sur une période de longueur 1. Par ailleurs,

$$|u|_{\mathrm{W}^{s,p}} = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|u(x)-u(x-h)|^p}{|h|^{n+sp}}\,\mathrm{d}h\mathrm{d}x.$$

Toute fonction $W^{s,p}((0,1)^n)$ peut être prolongée à une fonction périodique sur $(-1,1)^n$. Ainsi, l'hypothèse de périodicité ne restreint pas la généralité des résultats ou des preuves. Il faut préciser que les phases associées à des applications unimodulaires périodiques ne sont pas nécessairement périodiques.

Comme déjà mentionné, la preuve de l'existence d'un relèvement $\mathrm{W}^{s,p}$ dans le cas $s \in]0,1[$, $sp < 1$ est une preuve constructive. L'idée de cette construction est la suivante. Nous prenons d'abord des partitions dyadiques sur le tore $\mathbb{T}^n$,

$$\mathscr{P}_j \coloneqq \left\{ Q_j^m = \prod_{\ell+1}^{n} [m_\ell, m_\ell + 1[\;\middle|\; m_\ell \in [\![0, 2^j - 1]\!], \forall\, \ell \in [\![1,n]\!] \right\}, \qquad \forall\, j \in \mathbb{N}.$$

L'unique cube $Q_j \in \mathscr{P}_j$ qui contient l'élément $x \in \mathbb{T}^n$ est noté $Q_j(x)$. Ensuite, nous prenons la moyenne de u sur chacun des cubes Q_j :

$$u_j(x) \coloneqq \int_{Q_j(x)} u(y)\,\mathrm{d}y; \tag{2.8}$$

il est clair que les fonctions constantes par morceaux u_j approchent u dans L^p. Les applications $\dfrac{u_j}{|u_j|}$ (en utilisant la convention $0/0 = 1$) sont unimodulaires donc admettent des relèvements φ_j. En procédant par induction, les applications φ_j peuvent être choisies de façon qu'elles satisfassent une condition de la forme

$$(2.9) \qquad |\varphi_j - \varphi_{j-1}| \leqslant C\left(\frac{u_j}{|u_j|} - \frac{u_{j-1}}{|u_{j-1}|}\right).$$

A l'aide d'un résultat d'équivalence entre la semi-norme usuelle et une semi-norme dyadique que nous discutons dans la Section 2.5, (2.9) permet de conclure finalement que la suite φ_j converge dans L^p vers une application φ appartenant à $\mathrm{W}^{s,p}$, qui est alors un relèvement de u. Nous appelons cette méthode de construire une phase φ de u, la construction de Bourgain. Comme nous l'avons précisé, cette construction fournit déjà un contrôle de la semi-norme de la phase φ en fonction de la semi-norme de u :

$$(2.10) \qquad |\varphi|_{\mathrm{W}^{s,p}} \leqslant C(p,n)\frac{1}{s(1-sp)}|u|_{\mathrm{W}^{s,p}}.$$

Mais cette estimation n'est pas optimale.

Une méthode d'obtenir une meilleure estimation de $|\varphi|_{\mathrm{W}^{s,p}}$ est la méthode des moyennes, utilisée dans [**10**], et inspirée par [**27**]. Remarquons d'abord que le désavantage de la construction précédente est que, même si les partitions de $\mathbb{T}^n$ sont de plus en plus fines, les sommets des cubes dyadiques ont pour composantes toujours seulement les points de la forme $m2^{-j}$ avec $m \in \mathbb{N}$. Donc si u a une oscillation forte sur un voisinage d'un point irrationnel, une autre configuration des sommets de la partition $\mathscr{P}_j$ pourrait réduire les semi-normes des phases φ_j qui en résultent. Grâce à l'hypothèse de périodicité, on peut effectivement translater la partition initiale ; cette opération est équivalente à une translation de la fonction u. L'existence d'un vecteur de translation convenable y s'obtient par intégration en y sur $\mathbb{T}^n$. La méthode des moyennes consiste alors à effectuer la construction de la phase par la méthode de Bourgain pour toutes les translations $\tau_y u \coloneqq u(\cdot - y)$ de u, ce que fournit, pour chaque y fixé, une phase φ^y de $\tau_y u$. Nous avons le schéma suivant :

$$u \rightsquigarrow \tau_y u \rightsquigarrow \frac{\tau_y u}{|\tau_y u|} \rightsquigarrow \varphi_j^y \text{ phase de } \frac{\tau_y u}{|\tau_y u|} \text{ verifiant l'analogue à (2.9)}$$

$$\varphi_j^y \xrightarrow[j\to\infty]{} \varphi^y; \qquad \varphi^y \text{ phase de } \tau_y u \quad \Longrightarrow \quad \tau_{-y}\varphi^y \text{ phase de } u.$$

Jusqu'ici, la construction de la famille des phases $\varphi^y(\cdot + y)$ de u ne dépend pas des valeurs de p. Enfin, le but est d'estimer la moyenne de $|\varphi^y|^p_{\mathrm{W}^{s,p}}$ pour $y \in \mathbb{T}^n$. Le résultat obtenu est

$$(2.11) \qquad \int_{\mathbb{T}^n} |\varphi^y|^p_{\mathrm{W}^{s,p}}\,\mathrm{d}y \leqslant C(p,n)\frac{1}{s(1-sp)}|u|^p_{\mathrm{W}^{s,p}}.$$

Cela implique l'existence d'une phase φ de u qui vérifie (2.6) et qui améliore (2.10). La différence entre la preuve de (2.11) pour $p = 2$ donné par [**10**, Théorème E.1] et la preuve pour $p \geqslant 1$ de [**43**, Théorème 1.3] est que le cas $p = 2$ convient très bien aux calculs utilisant les séries de Fourier, mais, par contre, il n'est pas possible d'étendre la méthode au cas p quelconque. Nous obtenons le cas général par des nouveaux arguments non-L^2 qui s'avèrent plus simples que ceux initiaux. Un résultat-clef est un lemme ([**43**, Lemme 3.1]) qui permet d'estimer en moyenne (en fonction de la position de la grille dyadique) la différence entre une fonction et ses approximations par fonctions constantes par morceaux.

La méthode des moyennes s'est montrée utile dans l'étude de la relation entre les normes BMO et les normes BMO dyadiques ([**27**]). Aussi elle est utile dans la construction d'une phase optimale de $u\colon \mathbb{T}^n \to \mathbb{S}^1$. L'un des ingrédients de l'estimation de cette phase est une équivalence entre la semi-norme usuelle et des semi-normes dyadiques sur $\mathrm{W}^{s,p}$, résultat dû à Bourdaud (voir [**10**, Appendix A], [**9**]). Il est alors naturel de s'attendre à une possible amélioration des constantes dans les rélations de cette équivalence, en utilisant la méthode des moyennes. En effet, nous obtenons un résultat intéressant sur l'équivalence des semi-normes, présenté dans la Section 2.5.

2.3. L'optimalité des estimations dans le cas $p > 1$, $sp < 1$

L'estimation (2.6) (qui provient de (2.11)) est enfin une estimation optimale. Le sens de cette optimalité est qu'il existe un $u \in \mathrm{W}^{s,p}$ dont tous les relèvements satisfont l'inégalité

$$\frac{1}{(1-sp)^{1/p}}|u|_{\mathrm{W}^{s,p}} \leqslant C(p,n)|\varphi|_{\mathrm{W}^{s,p}} \qquad \text{(du moins si } 1-sp \ll 1\text{)},$$

qui est l'inégalité réciproque de (2.6) du point de vue $sp \nearrow 1$. L'existence d'une telle application u est obtenue en trouvant un exemple explicite : u est définie par

$$u \coloneqq \exp(\mathrm{i}\varphi_s),$$

pour un relèvement φ_s choisit convenablement. D'une part, u vérifie

$$c(p,n) \leqslant |u|_{\mathrm{W}^{s,p}} \leqslant C(n,p), \tag{2.12}$$

avec une preuve plutôt technique ([**43**, Lemma 8.14]). D'autre part, si $\varphi \in \mathrm{W}^{s,p}$ est une autre phase quelconque de u, elle vérifie

$$|\varphi|_{\mathrm{W}^{s,p}} \geqslant C(n,p)\frac{1}{(1-sp)^{1/p}}. \tag{2.13}$$

Notre preuve de (2.13) pour le cas $p > 1$ ([**43**, Théorème 1.5]) simplifie d'une certaine manière celle pour $p = 2$ dans [**11**]. Cela a été possible grâce au fait que [**11**, Théorème 2] n'utilise pas toute la force de l'inégalité qui décrit $\mathrm{W}^{1-\varepsilon}_0(]0,1[) \hookrightarrow \mathrm{L}^{1/\varepsilon}(]0,1[)$. En effet, il suffisait de l'appliquer seulement pour les fonctions indicatrices des sous-ensembles mesurables A :

$$|A|^{\varepsilon}|{}^{c}A|^{\varepsilon} \leqslant C\varepsilon \int_A \int_{{}^{c}A} \frac{1}{|x-y|^{2-\varepsilon}}\, \mathrm{d}x\mathrm{d}y.$$

Nous avons obtenu cette inégalité par une méthode alternative : en utilisant un théorème de réarrangement de Garsia et Rodemich ([**28**]). En particulier, ce théorème montre que le réarrangement décroissant d'une fonction défini sur un intervalle de $\mathbb{R}$ est aussi décroissant pour les semi-normes de Sobolev fractionnaires. Plus précisément, si f est une fonction définie sur l'intervalle $]0,1[$ et positive, et $f^*\colon]0,1[\to \mathbb{R}_+$ est le réarrangement décroissant usuel de f :

$$f^*(x) \coloneqq \inf\Big\{\lambda \in \mathbb{R} \;\Big|\; x \geqslant |\{t \in [0,1[\ ;\ \lambda < f(t)\}|\Big\},$$

alors par [**28**, Théorème I.1],

$$|f^*|_{W^{s,p}(]0,1[)} \leqslant |f|_{W^{s,p}(]0,1[)}.$$

Sur $\mathbb{R}$ entier, ce résultat est dû à Riesz (voir [**35**, section 3.3]).

Toutefois, même si la méthode des moyennes nous donne l'existence d'une phase qui satisfait une meilleure estimation que celle vérifiée *a priori* par une phase obtenue par la construction de Bourgain, il reste encore la question suivante. Est-ce que (2.10) peut être améliorée si nous considérons seulement les phases construites par cette méthode ? Nous nous entendons à une réponse négative à cette question. Le raison pour cela est qu'un ingrédient important dans la preuve de l'estimation (2.10) est une inégalité des semi-normes qui est optimale ([**10**, Lemma A.3]). Cependant, ce fait ne suffit pas pour conclure à l'optimalité du couple « construction de Bourgain – estimation (2.10) ». Nous aurions besoin d'un exemple d'application u et de phase φ construite par cette méthode qui satisfassent

$$\frac{1}{s(1-sp)}|u|_{W^{s,p}} \leqslant C(p,n)|\varphi|_{W^{s,p}};$$

mais nous n'avons pas encore un tel exemple.

2.4. Estimations optimales dans le cas $sp \geqslant 1$

Nous considérons maintenant $u \in W^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ avec $0 < s < \infty$, $1 \leqslant p < \infty$ et $sp \geqslant 1$. Nous rappelons que nous avons la propriété du relèvement dans les cas $0 < s < 1$, $sp \geqslant n$ ou $s \geqslant 1$, $sp \geqslant 2$. En plus, par la nature de la preuve de l'existence d'une phase φ, qui n'est pas constructive (contrairement au cas $sp < 1$), les estimations de la forme (2.3) doivent être intrinsèques à l'identité (2.1).

Si $s \geqslant 1$ est un entier, nous différentions (2.1) une fois et, à l'aide des inégalités de Gagliardo-Nirenberg et de l'inégalité de Poincaré, nous obtenons l'estimation linéaire

$$|\varphi|_{W^{s,p}} \leqslant C(s,p,n)|u|_{W^{s,p}}. \tag{2.14}$$

Le cas s non-entier est plus délicat et utilise les décompositions de Littlewood-Paley, mais l'estimation (2.14) reste valable.

Pour le cas $s = \dfrac{1}{p}$, comme nous l'avons déjà précisé, il n'y a pas de contrôle du type (2.3). Mais pour les autres cas restants, c'est-à-dire $n = 1$, $sp > 1$ ou $n \geqslant 2$, $sp \geqslant n$, il existe un contrôle de la forme

$$|\varphi|_{W^{s,p}} \leqslant C(s,p,n)\left(|u|_{W^{s,p}} + |u|^{1/s}_{W^{s,p}}\right). \tag{2.15}$$

L'estimation (2.15) est due à Merlet pour le cas uni-dimensionnel. Nous montrons qu'elle reste valable pour $n \geqslant 1$ quelconque. La preuve de Merlet ([**38**]) repose sur des arguments uni-dimensionnels ce qui requiert une nouvelle méthode pour démontrer (2.15) quand $n \geqslant 2$. Il s'agit de la méthode de la factorisation que nous présentons dans la Section 2.6 et qui a à la base une construction d'une fonction φ_1 qui est « presque » une phase de u. Ici est le seul endroit où les estimations pour $sp \geqslant 1$ sont obtenues d'une manière extrinsèque à (2.1).

Naturellement, la même méthode fournit une nouvelle preuve de (2.15) pour le cas $n = 1$. Revenus ainsi à (2.15) en dimension 1, nous y donnons encore deux preuves.

L'une est une simplification de l'argument de Merlet. L'autre est inspirée par le travail de Nguyen dans [**48**] et, contrairement à notre premier argument, est une preuve non constructive ; elle utilise la théorie de la dualité des espaces de fonctions, aussi que des propriétés de l'extension harmonique et de la fonction maximale.

Quant à la question d'optimalité des estimations (2.14) et (2.15), comparativement au cas $sp < 1$, le problème est beaucoup simplifié grâce au résultat suivant de [**10**]. Si $sp \geqslant 1$, les applications dans $\mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{Z})$ sont constantes. Cela implique l'unicité modulo 2π de la phase d'une application $u \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$. Donc, pour démontrer l'optimalité des estimations, il suffit de montrer qu'ils ne peuvent pas être améliorées pour un certain φ_0 et $u \coloneqq \exp(\mathrm{i}\varphi_0)$.

Par ailleurs, précisons ce que nous entendons par l'« optimalité » dans ce cas. Pour $s \geqslant 1$, vu que l'estimation (2.14) est linéaire, elle est dite optimale si

$$\frac{|\varphi|_{\mathrm{W}^{s,p}}}{|u|_{\mathrm{W}^{s,p}}} > 0 \quad \text{quand } |\varphi|_{\mathrm{W}^{s,p}} \to 0, \quad \text{ou quand } |u|_{\mathrm{W}^{s,p}} \to 0.$$

Pour le cas $s < 1$, l'optimalité concerne à nouveau le caractère linéaire de (2.15) quand $|u|_{\mathrm{W}^{s,p}}$ est petit, et, quand $|u|_{\mathrm{W}^{s,p}}$ est grand, l'optimalité de (2.15) se traduit en l'optimalité de la puissance $\dfrac{1}{s}$ dans le deuxième terme de (2.15).

2.5. Une équivalence des semi-normes de $\mathrm{W}^{s,p}$

Soit $f \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{C})$ avec $0 < s < 1$, $1 \leqslant p < \infty$ et $sp < 1$, et soient f_j les moyennes de f sur chaque cube dyadique (voir (2.8)). L'équivalence des semi-normes

$$(2.16)\quad \underbrace{\left(\int_{\mathrm{T}^n}\int_{\mathrm{T}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}}_{|f|^p_{\mathrm{W}^{s,p}} \eqqcolon X(f)} \sim \underbrace{\left(\sum_{j\geqslant 1} 2^{spj}\|f_j - f_{j-1}\|^p_{\mathrm{L}^p}\right)^{\frac{1}{p}}}_{\eqqcolon Y(f)} \sim \underbrace{\left(\sum_{j\geqslant 0} 2^{spj}\|f - f_j\|^p_{\mathrm{L}^p}\right)^{\frac{1}{p}}}_{\eqqcolon Z(f)}$$

est bien connue (voir [**10**, Théorème A.1]). Si nous mettons en évidence la dépendance de s des constantes d'équivalence dans (2.16), alors nous trouvons les formes quantitatives suivantes :

$$(2.17)\quad s^p Z \leqslant c_1(p,n)Y; \qquad Y \leqslant 2Z; \qquad Z \leqslant c_2(p,n)X$$

et

$$(2.18)\quad X \leqslant c_3(p,n)\frac{1}{s^p(1-sp)^p}Y.$$

([**43**, Lemma 8.3] ; voir aussi [**10**, preuve de Théorème A.1]). Seulement (2.18) a effectivement besoin de l'hypothèse $sp < 1$. Les inégalités de (2.17) sont valables indépendamment de cette condition et nous pouvons, de ce point de vue, identifier les semi-normes Y et Z.

Nous montrons que l'estimation (2.16) peut être améliorée en moyenne :

$$(2.19)\quad X(f) \leqslant c(p,n)\left[\frac{1}{s^2}\left(\frac{C(p)}{1-s}\right)^{\frac{1}{1-s}}\right]^p \int_{\mathbb{T}^n} Y(f^y)\,\mathrm{d}y$$

([**43**, Section 7]). La méthode utilisée est la méthode des moyennes vue dans la Section 2.2. La preuve est assez technique et utilise premièrement une approche discrète : ainsi, nous montrons d'abord, en dimension $n = 1$ que

$$\sum_{j\geqslant 1} 2^{spj} \left\|\tau_{2^{-j}} f - f\right\|^p_{\mathrm{L}^p(\mathbb{T})} \leqslant c(p) \left[\frac{1}{s^2}\left(\frac{C(p)}{1-s}\right)^{\frac{1}{1-s}}\right]^p \int_{\mathbb{T}} Y(f^y)\,\mathrm{d}y.$$

C'est ici que se trouve l'idée essentielle de l'estimation (et la difficulté de la preuve).

Un aspect important de (2.19) est qu'elle est valable sans aucune condition sur sp. Ce qui implique le fait que, en moyenne, l'équivalence des semi-normes (2.16) est valable pour tous $s \in]0,1[$ et $p \geqslant 1$.

En plus, (2.19) est aussi vraie quand $s = 1$ et $p = 2$. Cela peut inspirer une étude sur l'analogue de (2.18) pour $s \geqslant 1$.

Quant à l'optimalité de l'estimation (2.18), nous ne l'avons pas encore examinée. Toutefois, il est clair qu'elle n'est pas optimale dans le cas $p = 1$ car la constante dans (2.18) a une croissance exponentielle quand $s \nearrow 1$, plus rapide que celle de la constante dans (2.5) qui est juste de l'ordre de $\dfrac{1}{1-s}$.

2.6. La méthode de la factorisation – estimations optimales pour le cas $sp \geqslant 1$; construction d'une phase quand $sp < 1$

La méthode suivante de la factorisation d'une application à valeurs dans $\mathbb{S}^1$ est due à Mironescu ([**40**]). Nous présentons dans la suite les idées principales de la méthode ainsi que le façon dans lequel nous avons appliqué cette méthode dans [**43**] pour deux situations différentes.

Soit $u \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ avec $s \in]0,1[$, $p \in [1,\infty[$ tels que $sp \geqslant 1$. La méthode de la factorisation consiste en décomposer u comme un produit de la forme

$$u = \exp(\mathrm{i}\varphi_1)v \tag{2.20}$$

où $\varphi_1 \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{R})$ avec

$$|\varphi_1|_{\mathrm{W}^{s,p}} \leqslant C(p,n)|u|_{\mathrm{W}^{s,p}} \tag{2.21}$$

et $v \in \mathrm{W}^{sp,1}(\mathbb{T}^n;\mathbb{S}^1)$ avec

$$|v|_{\mathrm{W}^{sp,1}} \leqslant c(p,n)|u|^p_{\mathrm{W}^{s,p}}. \tag{2.22}$$

La construction de φ_1 est l'ingrédient central de la méthode. Elle se fait en plusieurs étapes. Nous prolongeons d'abord u à $\mathbb{R}^n$ entier d'une manière convenable (à l'aide de l'opérateur de prolongement habituel). L'extension de u qui en résulte (appelée encore u) vérifie, par ailleurs, que $u \in \mathrm{W}^{s,p}(\mathbb{R}^n;\mathbb{D}_2)\cap\mathrm{L}^\infty$, u restreint à une voisinage de 0 est à valeurs dans $\mathbb{S}^1$ et u est constant loin de 0. Puis nous étendons u à $\mathbb{R}^n\times]0,\infty[$ par convolution avec ρ_ε, une suite régularisante appropriée :

$$\omega(x,\varepsilon) \coloneqq u * \rho_\varepsilon(x).$$

Nous considérons ensuite une *presque*-projection sur le cercle : une fonction lisse Π qui est égale à la projection sur $\mathbb{S}^1$ sauf sur un petit disque centré en 0 :

$$\Pi \in \mathrm{C}^\infty(\mathbb{R}^2;\mathbb{R}^2), \qquad \Pi(z) \coloneqq \frac{z}{|z|} \quad \text{si} \quad z \in \mathbb{R}^2 \smallsetminus \mathbb{D}_{1/2}.$$

Soit ν la *projection approximative de* ω sur $\mathbb{S}^1$:

$$\nu(x,\varepsilon) \coloneqq \Pi \circ \omega(x,\varepsilon).$$

Finalement, l'application φ_1 est définie par

$$\varphi_1(x) \coloneqq \int_0^\infty \det\left(\frac{\partial \nu}{\partial \varepsilon}, \nu\right)(x,\varepsilon)\,\mathrm{d}\varepsilon,$$

où l'intégrale est convergente presque pour tout $x \in \mathbb{T}^n$. Nous pouvons dire que φ_1 représente une phase de la partie de u qui a des oscillations de petite amplitude, et cela dans le sens que, si les valeurs de u se trouvent toutes dans une bande suffisamment fine autour de $\mathbb{S}^1$ alors φ_1 est précisément une phase de u. En effet, soit un tel u. Alors ω a lui-aussi des valeurs proches de $\mathbb{S}^1$ et donc ν est une application (lisse) à valeurs exactement dans $\mathbb{S}^1$. Il résulte donc que $\nu = (\nu_1, \nu_2)$ admet une phase (lisse) : $\nu = \exp(\mathrm{i}\psi)$. En différentiant cette identité par rapport à ε, nous avons

$$\frac{\partial \nu}{\partial \varepsilon} = \mathrm{i}\frac{\partial \psi}{\partial \varepsilon}\nu \quad \Longrightarrow \quad \frac{\partial \psi}{\partial \varepsilon} = \nu_1 \frac{\partial \nu_2}{\partial \varepsilon} - \nu_2 \frac{\partial \nu_1}{\partial \varepsilon} = \det\left(\nu, \frac{\partial \nu}{\partial \varepsilon}\right).$$

Supposons qu'en intégrant la dernière identité pour $\varepsilon \in]0,\infty[$ nous obtenons une intégrale convergente – alors, en passant à l'exponentielle, nous trouvons que

$$\exp(\mathrm{i}\varphi_1(x)) = \lim_{\varepsilon\to 0} \nu(x,\varepsilon)\left(\lim_{\varepsilon\to\infty} \nu(x,\varepsilon)\right)^{-1}.$$

Comme les valeurs de u sont toujours de module proche de 1, alors le deuxième terme est égale à 1 et

$$\exp(\mathrm{i}\varphi_1(x)) = u(x),$$

presque pour tout $x \in \mathbb{T}^n$, par le théorème de différentiation de Lebesgue.

En plus, $\varphi_1 \in \mathrm{W}^{s,p}$, $v \coloneqq u\exp(-\mathrm{i}\varphi_1) \in \mathrm{W}^{1,sp}$ et les deux vérifient les estimations (2.21) et (2.22) ([**40**]). A l'aide de la factorisation (2.20), nous pouvons obtenir maintenant facilement l'estimation (2.15) pour $n \geqslant 2$, $sp \geqslant n$. En effet, comme $v \in \mathrm{W}^{1,sp}(\mathbb{T}^n)$ et $sp \geqslant 2$, alors v admet une phase $\varphi_2 \in \mathrm{W}^{1,sp}(\mathbb{T}^n)$ qui appartient ainsi aussi à $\mathrm{W}^{s,p}(\mathbb{T}^n)$. Donc $\varphi_1 + \varphi_2$ est l'unique phase (modulo 2π) de u. Enfin, (2.21) et (2.22) impliquent (2.15). D'une manière similaire, (2.15) peut être obtenu aussi dans le cas $n = 1$, $sp > 1$.

Une deuxième application de la méthode de la factorisation intervient dans le cadre $sp < 1$. Il faut noter que la construction précédente de φ_1 ne dépend pas des valeurs de s et de p, et l'estimation (2.21) est valable aussi dans le cas $sp < 1$ (seulement (2.22) nécessitait la condition $sp \geqslant 1$). Nous montrons cette fois-ci que

$$v \in \mathrm{W}^{sp,1}(\mathbb{T}^n;\mathbb{S}^1). \tag{2.23}$$

Comme les preuves de (2.21) et (2.22), la preuve de (2.23) n'est pas facile ; elle utilise l'extension harmonique, des inclusions de Sobolev et la théorie des espaces de Sobolev à poids. Par la preuve de (2.7), v admet une phase $\varphi_2 \in \mathrm{W}^{sp,1}$ qui est aussi bornée. Cela implique le fait que φ appartient aussi à $\mathrm{W}^{s,p}$ (par les inégalités de Gagliardo-Nirenberg) et donc $\varphi_1 + \varphi_2$ est une phase de u.

2.7. Preuves des résultats

Cette section contient le travail en collaboration avec P. Mironescu, qui fait objet d'une publication acceptée dans *Ann. Inst. H. Poincaré Anal. Non Linéaire* en avril 2014 ([**43**]).

Phases of unimodular complex valued maps: optimal estimates, the factorization method, and the sum-intersection property of Sobolev spaces

en collaboration avec P. Mironescu

Abstract: *We address and answer the question of optimal lifting estimates for unimodular complex valued maps: given $s > 0$ and $1 \leqslant p < \infty$, find the best possible estimate of the form $|\varphi|_{W^{s,p}} \lesssim F\left(|\exp(i\varphi)|_{W^{s,p}}\right)$.*

The most delicate case is $sp < 1$. In this case, we extend the results obtained by J. Bourgain, H. Brezis and P. Mironescu (2000, 2002) for $p = 2$ (using L^2 Fourier analysis and optimal constants in the Sobolev embeddings) by developing non L^2 estimates and an approach based on symmetrization. Following an idea of Bourgain, our proof also relies on averaged estimates for martingales. As a byproduct of our arguments, we obtain a characterization of fractional Sobolev spaces with $0 < s < 1$ involving averaged martingale estimates.

Also when $sp < 1$, we propose a new phase construction method, based on oscillations detection, and discuss existence of a bounded phase.

When $sp \geqslant 1$, we extend to higher dimensions a result on optimal estimates of Merlet (2006), based on one dimensional arguments. This extension requires new ingredients (factorization techniques, duality methods).

2.7.1. Introduction. — Our motivation is provided by the following problem:

Lifting estimate question. Let $\Omega \subset \mathbb{R}^n$ be smooth bounded simply connected, and let $0 < s < \infty$, $1 \leqslant p < \infty$. Assume that $W^{s,p}(\Omega; \mathbb{S}^1)$ has the *lifting property*, i.e., that every $u \in W^{s,p}(\Omega; \mathbb{S}^1)$ has a phase $\varphi \in W^{s,p}(\Omega; \mathbb{R})$. Which is the best possible estimate of the form

$$(2.24)\qquad |\varphi|_{W^{s,p}} \lesssim F(|u|_{W^{s,p}})?$$

Here, $A \lesssim B$ means $A \leqslant CB$, with C possibly depending on p and on the space dimension n, but not on s or u.

Estimate (2.24) can be seen as a reverse estimate for superposition operators. Superposition operators are mappings of the form

$$T_\Phi(\varphi) = \Phi \circ \varphi, \quad \forall\, \varphi \in X,$$

with X a function space. Classical questions concerning such operators are: under which regularity assumptions on Φ we have $T_\Phi : X \to X$, and existence of estimates of the form

$$(2.25)\qquad \|T_\Phi(\varphi)\|_X \leqslant G(\|\varphi\|_X);$$

see e.g. [**51**] for a detailed account of these topics. The questions we discuss in the present chapter are related to a sort of converse of (2.25), namely existence of estimates of the form

$$\|\varphi\|_X \leqslant F(\|T_\Phi(\varphi)\|_X) \tag{2.26}$$

(or of a similar estimate where the full norm $\|\cdot\|_X$ is replaced by a semi-norm $|\cdot|_X$). Clearly, (2.26) cannot hold for every Φ, even smooth (take $\Phi = 0$). A hint is given by the analysis of the case where $X = \mathrm{W}^{1,p}$. The fact that

$$\|\nabla(\Phi \circ \varphi)\|_{\mathrm{L}^p} = \|\Phi'(\varphi)\nabla\varphi\|_{\mathrm{L}^p}$$

suggests that, in order to have both (2.25) and (2.26), a reasonable condition is that

$$0 < a \leqslant |\Phi'| \leqslant b < \infty.$$

This suggests considering the *model nonlinearity* $\Phi(t) = \exp(\mathrm{i}t)$, which satisfies $|\Phi'| = 1$, and then the corresponding problem is given by (2.24).

For simplicity, we consider only periodic maps $u\colon \mathbb{T}^n \to \mathbb{S}^1$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (but it will be transparent from the proofs that the constructions and arguments we present extend to maps defined on Lipschitz bounded domains). If $u\colon \mathbb{T}^n \to \mathbb{S}^1$ is smooth, then u has a smooth phase $\varphi\colon [0,1)^n \to \mathbb{R}$. Of course, such a phase need not be $\mathbb{Z}^n$-periodic and thus cannot be identified with a smooth map on $\mathbb{T}^n$. However, for notational simplicity, we still write most of the times $\varphi\colon \mathbb{T}^n \to \mathbb{R}$.

The maps we consider are normed in the standard way (over a period); e.g., we let $\|f\|_{\mathrm{L}^p} \coloneqq \|f\|_{\mathrm{L}^p([0,1)^n)}$.

Before presenting our contribution, let us briefly recall some previously known results concerning the existence of phases $\varphi\colon [0,1)^n \to \mathbb{R}$ of maps $u\colon \mathbb{T}^n \to \mathbb{S}^1$, and the corresponding estimates. First, the characterization of s and p such that $\mathrm{W}^{s,p}(\Omega;\mathbb{S}^1)$ has the lifting property was obtained in [**10**] and is the following.

Theorem 2.1 ([**10**]). — *The space* $\mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ *has the lifting property precisely in the following cases:*

1. $sp < 1$.
2. $sp \geqslant n$.
3. $s \geqslant 1$ *and* $sp \geqslant 2$.

Concerning optimal estimates of the form (2.24), two qualitatively different situations are to be considered. As an illustration, let us assume that we have an estimate of the form (2.24) at our disposal, and also that the equality $|\varphi_0|_{\mathrm{W}^{s,p}} = F\left(|u|_{\mathrm{W}^{s,p}}\right)$ holds for some $\varphi_0 \in \mathrm{W}^{s,p}$, with $u \coloneqq \exp(\mathrm{i}\varphi_0)$. Starting from this, we would like to assert that (2.24) is optimal. This is easily obtained when $sp \geqslant 1$. Indeed, in this case, if $u = \exp(\mathrm{i}\varphi_1) = \exp(\mathrm{i}\varphi_2)$ with $\varphi_1, \varphi_2 \in \mathrm{W}^{s,p}$, then $\varphi_1 = \varphi_2 \pmod{2\pi}$ [**10**, Theorem B.1]; thus the phase (if it exists) is unique. Consequently, there is *no phase* $\varphi \in \mathrm{W}^{s,p}$ of u such that $|\varphi|_{\mathrm{W}^{s,p}} < F\left(|u|_{\mathrm{W}^{s,p}}\right)$, and thus (2.24) is optimal. We will present in Subsection 2.7.4 the optimal estimates corresponding to the range $sp \geqslant 1$; for the time being let us only mention the strategy. First, an inspection of the construction of phases in [**10**] and [**40**] leads to estimates of the form (2.24). Next, we test these estimates on typical $\mathrm{W}^{s,p}$ functions (like $x \mapsto |x|^{-\alpha}$, with $(\alpha + s)p < n$) and conclude to their optimality. (Special cases of the results in Subsection 2.7.4 were obtained by Merlet [**38**].)

Much more involved is the case where $sp < 1$. Indeed, assume that we have established an estimate of the type (2.24) and that we want to prove its optimality. This time, if φ is a $\mathrm{W}^{s,p}$ phase of u, then so is $\varphi + 2\pi \mathbb{1}_A$, with A a smooth compact subset of Ω. Thus even if the estimate (2.24) cannot be improved for a *specific* φ, it could be possible to obtain *another phase* of u satisfying a *better estimate*.

Optimality when $sp < 1$ and $p = 2$ was investigated in [**10**] and [**11**]; the corresponding optimal estimates have implications in the analysis of the Ginzburg-Landau equation ([**12**]) and were part of the original motivation in studying (2.24). In order to explain the results obtained in [**10**], [**11**], we first recall a phase construction method due to Bourgain and presented in [**10**]. Assume that $sp < 1$ and let $u \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$. For $j \in \mathbb{N}$, we let $\mathscr{P}_j$ denote the set of the (dyadic) cubes of the form $2^{-j} \prod_{l=1}^{n} [m_l, m_l + 1)$, with $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Thus each $x \in \mathbb{R}^n$ belongs to exactly one cube $Q_j(x) \in \mathscr{P}_j$, and we have $Q_j(x) \subset Q_{j-1}(x)$ if $j \geqslant 1$. If $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, then we let

$$u_j(x) \coloneqq E_j u(x) \quad \text{denote the average of } u \text{ on } Q_j(x). \tag{2.27}$$

We let $\mathscr{E}_j$ denote the set of functions which are constant on every cube of $\mathscr{P}_j$. For a given $u \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$, the construction of a phase φ goes as follows. Let u_j be as in (2.27), and set $U^j \coloneqq \dfrac{u_j}{|u_j|} \in \mathscr{E}_j$, with the convention $\dfrac{0}{0} = 1$. We then let φ^0 be any real number such that $U^0 = \exp(\mathrm{i}\varphi^0)$ and next construct inductively a phase $\varphi^j \in \mathscr{E}_j$ of U^j such that

$$|\varphi^j - \varphi^{j-1}| \lesssim |U^j - U^{j-1}|. \tag{2.28}$$

The arguments developed in [**10**] imply that the sequence $(\varphi^j)_j$ converges in L^p to a phase φ of u satisfying the estimate (2.29) below.

Theorem 2.2 ([**10**]). — *Assume that $sp < 1$. Then every $u \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$ has a phase $\varphi \in \mathrm{W}^{s,p}$ satisfying*

$$|\varphi|^p_{\mathrm{W}^{s,p}} \lesssim \frac{1}{s^p(1-sp)^p} |u|^p_{\mathrm{W}^{s,p}}. \tag{2.29}$$

Here, $|\cdot|_{\mathrm{W}^{s,p}}$ is the standard Gagliardo semi-norm,

$$|f|^p_{\mathrm{W}^{s,p}} = \int\int \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y.$$

As explained above, when $sp < 1$ the phase is not unique, and this raises the question of the optimality of (2.29). It turns out that (2.29) is not optimal. (It is proved in [**10**, Appendix A] that the estimates used in the proof of Theorem 2.2 are essentially optimal and thus cannot lead to an estimate better than (2.29). However, this does not imply that the phase obtained via the iterative construction in formula (2.28) does not satisfy an improved estimate. We do not have an example of u such that the corresponding φ does not satisfy (2.30).) When $p > 1$, an improved estimate is provided by the following result.

Theorem 2.3. — *Let $0 < s < 1$ and $1 \leqslant p < \infty$ be such that $sp < 1$. Let $u \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$. Then there exists a phase φ of u satisfying the estimate*

$$|\varphi|^p_{\mathrm{W}^{s,p}} \lesssim \frac{1}{s^p(1-sp)} |u|^p_{\mathrm{W}^{s,p}}. \tag{2.30}$$

When $p = 2$, the above result is due to Bourgain ([**10**, Theorem 3.1]). Bourgain's proof relies on an averaging method, reminiscent of Garnett and Jones ([**27**]). The idea is to perform the dyadic construction explained above starting from $u^y := u(\cdot - y)$ instead of u, and obtain a corresponding phase φ^y. Then prove that, for some $y \in \mathbb{T}^n$, $\varphi^y(\cdot + y)$ (which is clearly a phase of u) satisfies the improved estimate (2.30). While the first part of the proof (construction of φ^y) does not depend on p, the argument leading to the last part (existence of an appropriate y) in [**10**] is based on L^2 Fourier analysis. Thus, in the proof of Theorem 2.3, our main task was to develop new, non L^2, arguments.

We continue with a digression related to the use of the averaging method. In [**10**], the proof of (2.29) (and of the corresponding phase existence result) is based on the semi-norm equivalence ([**10**, Theorem A.1])

$$|f|^p_{\mathrm{W}^{s,p}} \sim \sum_{j\geqslant 1} 2^{spj} \|f_j - f_{j-1}\|^p_{\mathrm{L}^p}. \tag{2.31}$$

Here, the averages f_j are as in (2.27) (with u replaced by f). It is easy to see that the above semi-norm equivalence *cannot hold* when $sp \geqslant 1$. Indeed, let $0 < s < 1$ and $1 \leqslant p < \infty$ be such that $sp \geqslant 1$. Let f be (the periodic extension of) the characteristic function of $\left[0, \frac{1}{2}\right)^n$. Then the right-hand side of (2.31) is finite (since $f_j = f$, $\forall\, j \geqslant 1$) but $f \notin \mathrm{W}^{s,p}$, as one may easily check. However, we have the following result, proving that the semi-norm equivalence (2.31) is valid *in average* when $0 < s < 1$, irrespective of the assumption $sp < 1$.

Theorem 2.4. — *Let $0 < s < 1$ and $1 \leqslant p < \infty$. Let $f^y(x) := f(x - y)$. Then we have*

$$|f|^p_{\mathrm{W}^{s,p}} \sim \int_{\mathbb{T}^n} \sum_{j\geqslant 1} 2^{spj} \left\|(f^y)_j - (f^y)_{j-1}\right\|^p_{\mathrm{L}^p} \,\mathrm{d}y. \tag{2.32}$$

This leads to the following picture, reminiscent of the connection discovered in [**27**] between BMO and dyadic BMO semi-norms:

1. The dyadic semi-norm $\left(\sum_{j\geqslant 1} 2^{sjp} \|f_j - f_{j-1}\|^p_{\mathrm{L}^p}\right)^{1/p}$ is equivalent to the standard semi-norm $|\cdot|_{\mathrm{W}^{s,p}}$ precisely when $sp < 1$. This is Bourdaud's result [**9**, Théorème 5]. We note that this equivalence requires $0 < s < 1$, and for such s it holds only *for some* p's in the range $[1, \infty)$.
2. However, in average, the two semi-norms are equivalent in the *full range* $0 < s < 1$, $1 \leqslant p < \infty$.

We next turn to the question of the optimality of the estimate (2.29), settled in [**10**, Remark 7] for $p = 2$ and $n \geqslant 2$, and in [**11**, Theorem 2] for $p = 2$ and $n = 1$.

Theorem 2.5. — *Assume that $1 < p < \infty$. Then estimate* (2.30) *is optimal.*

Here, optimality means that (2.30) cannot be improved to

$$|\varphi|^p_{\mathrm{W}^{s,p}} \leqslant \frac{\varepsilon(s)}{s^p(1 - sp)} |u|^p_{\mathrm{W}^{s,p}},$$

with $\varepsilon(s) \to 0$ as $sp \nearrow 1$.

The original argument in [**11**, Theorem 2] relies on an involved result: the behavior of the best constant in the embedding $\mathrm{W}^{1-\varepsilon,1}((0,1)) \hookrightarrow \mathrm{L}^{1/\varepsilon}((0,1))$. We develop here a related, but simpler, argument, whose main ingredient is the fact that the nonincreasing rearrangement on an interval does not increase the fractional Sobolev norms. This is well-known on the real line, and goes back to Riesz when $p = 2$ ([**35**, Lemma 3.6]); on an interval, the corresponding result is more recent and is due to Garsia and Rodemich ([**28**]).

As it turns out, the proofs of Theorems 2.3 and 2.5 we present below are slightly simpler than the original ones even when $p = 2$.

The reader may wonder about the role of the assumption $p > 1$ in Theorem 2.5. It turns out that this result is wrong when $p = 1$. Instead, we have the following improved estimate.

Proposition 2.6. — *Let* $0 < s < 1$. *Then every map* $u \in \mathrm{W}^{s,1}(\mathbb{T}^n; \mathbb{S}^1)$ *has a phase* φ *such that*

$$|\varphi|_{\mathrm{W}^{s,1}} \leqslant 2|u|_{\mathrm{W}^{s,1}}. \tag{2.33}$$

Estimate (2.33) is essentially optimal, since we clearly have $|u|_{\mathrm{W}^{s,1}} \leqslant |\varphi|_{\mathrm{W}^{s,1}}$. The proof of Proposition 2.6 follows the approach of Dávila and Ignat ([**23**]), who established, for BV maps $u\colon \mathbb{T}^n \to \mathbb{S}^1$, the existence of a BV phase φ satisfying the (optimal) estimate $|\varphi|_{\mathrm{BV}} \leqslant 2|u|_{\mathrm{BV}}$.

This section is organized as follows. Subsection 2.7.2, 2.7.3 and 2.7.4 are devoted to optimal estimates. In Subsection 2.7.2, we prove Theorem 2.3, which leads to an optimal estimate when $sp < 1$ and $p > 1$, and Proposition 2.6, giving an optimal estimate when $s < 1$ and $p = 1$. Subsection 2.7.3 contains the proof of Theorem 2.5, which asserts the optimality of the estimate in Theorem 2.3. In Subsection 2.7.4, we examine optimal estimates when $sp \geqslant 1$.

Subsections 2.7.5 and 2.7.6 are devoted to further developments. In Subsection 2.7.5.1 we discuss the existence of a bounded phase when $sp < 1$. In Subsection 2.7.5.2, we describe a new method for constructing phases when $sp < 1$. This construction combines a factorization technique developed by the first author [**41**], [**42**] with an averaging idea due to Dávila and Ignat [**23**]. Subsection 2.7.6 is devoted to the proof of Theorem 2.4.

The final Subsection 2.7.7 gathers various useful auxiliary estimates.

2.7.2. Optimal estimates when $sp < 1$. Proof of Theorem 2.3. — We start with some preliminary results. We recall that $Q_j(x)$ is the unique cube in $\mathscr{P}_j$ such that $x \in Q_j(x)$. We set $f_j(x) := \fint_{Q_j(x)} f(z)\,\mathrm{d}z$, $\tau_h f(x) := f(x-h)$, and we associate with f, s and p the following quantities:

$$X(f) := |f|^p_{\mathrm{W}^{s,p}} = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|f(x)-\tau_h f(x)|^p}{|h|^{n+sp}}\,\mathrm{d}x\mathrm{d}h, \tag{2.34}$$

$$Y(f) := \sum_{j\geqslant 1} 2^{spj}\|f_j - f_{j-1}\|^p_{\mathrm{L}^p}, \tag{2.35}$$

$$Z(f) := \sum_{j\geqslant 0} 2^{spj}\|f - f_j\|^p_{\mathrm{L}^p}. \tag{2.36}$$

When $sp < 1$, we have that $X(f)$, $Y(f)$ and $Z(f)$ are equivalent semi-norms in $\mathrm{W}^{s,p}(\mathbb{T}^n)$. This fact was established by Bourdaud ([**9**]); see [**10**, Theorem A.1] for a quantitative form of this equivalence. We briefly recall in Subsection 2.7.7.1 the result in [**10**] with a slightly different proof; see Lemma 2.27.

It can be easily shown that the phases φ^j given by (2.28) satisfy the following inequality ([**10**, (1.5)]):

$$|\varphi^j - \varphi^{j-1}| \lesssim |u - u_j| + |u - u_{j-1}|, \quad \forall\, j \geqslant 1. \tag{2.37}$$

In [**10**], estimate (2.29) is obtained by combining (2.37) with the (quantitative form of) the equivalence between $X(u)$, $Y(u)$ and $Z(u)$ (with X, Y and Z as in (2.34) – (2.36)).

The proof of the improved estimate (2.30) is more subtle. In order to obtain (2.30), we follow the approach in [**10**], which is itself inspired by a result of Garnett and Jones ([**27**]) showing that one can recover the standard BMO norm of a function u from the dyadic BMO norm of a suitable translation $\tau_h u$ of u. More specifically, the argument goes as follows. Let $u^y \coloneqq \tau_y u$ and let φ^y be the phase of u^y obtained via Bourgain's construction, i.e., $\varphi^y \coloneqq \lim_{j\to\infty} \varphi^{y,j}$. Here, $\varphi^{y,j} \in \mathscr{E}_j$ is a phase of $\dfrac{u^y_j}{|u^y_j|}$ satisfying

$$|\varphi^{y,j} - \varphi^{y,j-1}| \lesssim |u^y - u^y_j| + |u^y - u^y_{j-1}|, \quad \forall\, j \geqslant 1. \tag{2.38}$$

In the spirit of [**10**], we will prove that

$$\int_{\mathbb{T}^n} |\varphi^y|^p_{W^{s,p}}\, dy \lesssim \frac{1}{s^p(1-sp)} |u|^p_{W^{s,p}}. \tag{2.39}$$

Indeed, for every measurable function $f\colon \mathbb{T}^n \to \mathbb{C}$ we clearly have

$$\begin{aligned} |f|^p_{W^{s,p}} &= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|(\tau_h - \mathrm{id})f(x)|^p}{|h|^{n+sp}}\, dx dh \\ &\leqslant \sum_{j\geqslant 0} 2^{(n+sp)(j+1)} \int_{|h|\in I_j}\int_{\mathbb{T}^n} |(\tau_h - \mathrm{id})f(x)|^p\, dx dh, \end{aligned}$$

where $I_j \coloneqq \Big[\dfrac{1}{2^{j+1}}, \dfrac{1}{2^j}\Big)$. We find that the average of $|\varphi^y|^p_{W^{s,p}}$ can be estimated by

$$\int_{\mathbb{T}^n} |\varphi^y|^p_{W^{s,p}}\, dy \leqslant \int_{\mathbb{T}^n} \sum_{j\geqslant 0} 2^{(n+sp)(j+1)} \int_{|h|\in I_j}\int_{\mathbb{T}^n} |(\tau_h - \mathrm{id})\varphi^y|^p\, dx dh dy. \tag{2.40}$$

In order to estimate the right-hand side of (2.40), we start from

$$|(\tau_h - \mathrm{id})\varphi^y| \leqslant \left|(\tau_h - \mathrm{id})\varphi^{y,j}\right| + \left|(\tau_h - \mathrm{id})\left(\varphi^y - \varphi^{y,j}\right)\right|, \quad \forall\, j \geqslant 0. \tag{2.41}$$

Consider now $\rho \coloneqq \mathbb{1}_{(-\frac{1}{2},\frac{1}{2})^n}$, and set $\rho_\varepsilon(x) \coloneqq \dfrac{1}{\varepsilon^n}\rho\left(\dfrac{x}{\varepsilon}\right)$, $\forall\, \varepsilon > 0$, $\forall\, x$. We define

$$A_{k,j} \coloneqq \left\{ x \in \mathbb{T}^n \;\middle|\; \mathrm{dist}(x, \partial Q) \leqslant \frac{1}{2^j} \text{ for some } Q \in \mathscr{P}_k \right\}.$$

By Lemma 2.30 in Subsection 2.7.7.2, when $|h| \in I_j$ we have

$$\begin{aligned} \left|(\tau_h - \mathrm{id})\varphi^{y,j}\right| &= \left|(\tau_h - \mathrm{id})\left(\varphi^{y,j} - \varphi^{y,0}\right)\right| = \left|\sum_{k=1}^{j} (\tau_h - \mathrm{id})\left(\varphi^{y,k} - \varphi^{y,k-1}\right)\right| \\ &\leqslant \sum_{k=1}^{j} \left|(\tau_h - \mathrm{id})\left(\varphi^{y,k} - \varphi^{y,k-1}\right)\right| \lesssim \sum_{k=1}^{j} \left|\varphi^{y,k} - \varphi^{y,k-1}\right| * \rho_{2^{2-k}} \mathbb{1}_{A_{k,j}}. \end{aligned} \tag{2.42}$$

Before going further, let us note that

$$\rho_{2^{2-k}} \lesssim \rho_{2^{3-k}} \qquad \text{and} \qquad A_{k,j} \subset A_{k+1,j}. \tag{2.43}$$

By (2.38), (2.42) and (2.43), we thus have

$$\left|(\tau_h - \mathrm{id})\varphi^{y,j}\right| \lesssim \sum_{k=0}^{j} \left|u^y - u_k^y\right| * \rho_{2^{3-k}} \mathbb{1}_{A_{k+1,j}}.$$

Hence

$$\left|(\tau_h - \mathrm{id})\varphi^{y,j}(x)\right|^p \lesssim \left(\sum_{k=0}^{j} \left|u^y - u_k^y\right| * \rho_{2^{3-k}} \mathbb{1}_{A_{k+1,j}}(x)\right)^p =: J_{1,j}(x,y). \tag{2.44}$$

On the other hand, (2.38) implies

$$\begin{aligned} \left\|(\tau_h - \mathrm{id})\left(\varphi^y - \varphi^{y,j}\right)\right\|_{\mathrm{L}^p}^p \lesssim \left\|\varphi^y - \varphi^{y,j}\right\|_{\mathrm{L}^p}^p &\leqslant \int_{\mathbb{T}^n} \left(\sum_{k\geqslant j+1} \left|\varphi^{y,k}(x) - \varphi^{y,k-1}(x)\right|\right)^p \mathrm{d}x \\ &\lesssim \int_{\mathbb{T}^n} \left(\sum_{k\geqslant j} \left|u^y(x) - u_k^y(x)\right|\right)^p \mathrm{d}x =: J_{2,j}(y). \end{aligned} \tag{2.45}$$

By combining the estimates (2.44) and (2.45) with (2.40) and (2.41), we find that

$$\int_{\mathbb{T}^n} |\varphi^y|_{\mathrm{W}^{s,p}}^p \,\mathrm{d}y \lesssim \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \sum_{j\geqslant 0} 2^{spj} J_{1,j}(x,y)\,\mathrm{d}y\mathrm{d}x + \int_{\mathbb{T}^n} \sum_{j\geqslant 0} 2^{spj} J_{2,j}(y)\,\mathrm{d}y =: L_1 + L_2.$$

We first estimate the term L_2 *via* a Schur type estimate (Corollary 2.26) and Lemma 2.27:

$$\begin{aligned} \sum_{j\geqslant 0} 2^{spj} J_{2,j}(y) &= \int_{\mathbb{T}^n} \sum_{j\geqslant 0} \left(\sum_{k\geqslant j} 2^{s(j-k)}\left(2^{sk}\left|u^y(x) - u_k^y(x)\right|\right)\right)^p \mathrm{d}x \\ &\lesssim \frac{1}{s^p}\sum_{k\geqslant 0} 2^{spk}\left\|u^y - u_k^y\right\|_{\mathrm{L}^p}^p = \frac{1}{s^p} Z\left(u^y\right) \lesssim \frac{1}{s^p} X\left(u^y\right) = \frac{1}{s^p}|u|_{\mathrm{W}^{s,p}}^p, \end{aligned}$$

for all $y \in \mathbb{T}^n$. Consequently,

$$L_2 \lesssim \frac{1}{s^p}|u|_{\mathrm{W}^{s,p}}^p. \tag{2.46}$$

(As in [**10**], the integration with respect to y does not play any role in the estimate satisfied by L_2.)

We now turn to L_1. We decompose the sets $A_{k,j}$, which are increasing with k, as a finite disjoint union of sets by defining

$$B_{k,j} := A_{k,j} \smallsetminus A_{k-1,j}, \quad \forall\, k \geqslant 2 \qquad \text{and} \qquad B_{1,j} := A_{1,j}.$$

Thus, $A_{k,j} = \bigsqcup_{1\leqslant t\leqslant k} B_{t,j}$ and we have

$$\begin{aligned} L_1 &= \sum_{j\geqslant 0} 2^{spj} \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left(\sum_{k=0}^{j}\sum_{t=1}^{k+1} \left|u^y - u_k^y\right| * \rho_{2^{4-k}} \mathbb{1}_{B_{t,j}}(x)\right)^p \mathrm{d}y\mathrm{d}x \\ &= \sum_{j\geqslant 0} 2^{spj} \sum_{t=1}^{j+1} \int_{B_{t,j}} \left\|\sum_{k=t-1}^{j} \left|u^y - u_k^y\right| * \rho_{2^{4-k}}(x)\right\|_{\mathrm{L}_y^p(\mathbb{T}^n)}^p \mathrm{d}x. \end{aligned}$$

Using Minkowski's inequality and noting that $|B_{t,j}| \leqslant |A_{t,j}| \lesssim 2^{t-j}$, we find

$$L_1 \lesssim \sum_{j\geqslant 0} 2^{spj} \sum_{t=1}^{j+1} 2^{t-j} \left(\sum_{k=t-1}^{j} \sup_x \left\|\left|u^y - u_k^y\right| * \rho_{2^{4-k}}(x)\right\|_{\mathrm{L}_y^p(\mathbb{T}^n)}\right)^p.$$

Now comes the key estimate.

Lemma 2.7. — *Assume that* $0 < s < 1$. *Let* $u \in \mathrm{W}^{s,p}(\mathbb{T}^n)$, *and define* $g_k \colon \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{R}$ *by*

$$g_k(x,y) := \left|u^y - u^y_k\right| * \rho_{2^{4-k}}(x).$$

Then $\sum_{k\geqslant 0} a_k^p \leqslant 2\,|u|^p_{\mathrm{W}^{s,p}}$, *where*

$$a_k := 2^{sk} \sup_x \|g_k(x,\cdot)\|_{\mathrm{L}^p}, \quad \forall\, k \geqslant 0.$$

Proof. — Hölder's inequality combined with the fact that the integral of ρ equals 1 gives

$$\int_{\mathbb{T}^n} |g_k(x,y)|^p \,\mathrm{d}y \leqslant \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left|u^y - u^y_k\right|^p (x-z)\rho_{2^{4-k}}(z)\,\mathrm{d}y\mathrm{d}z. \tag{2.47}$$

We next note that

$$\begin{aligned}\left|u^y - u^y_k\right|^p (x-z) &= \left|\fint_{Q_k(x-z)} \left(u^y(x-z) - u^y(w)\right)\mathrm{d}w\right|^p \\ &\leqslant 2^{nk}\int_{B(x-z,2^{-k})} |u^y(x-z) - u^y(w)|^p \,\mathrm{d}w;\end{aligned} \tag{2.48}$$

here, we use Hölder's inequality together with the fact that $Q_k(x-z) \subset B\left(x-z, 2^{-k}\right)$.

Integration of (2.48) over y leads to

$$\begin{aligned}\int_{\mathbb{T}^n} \left|u^y - u^y_k\right|^p (x-z)\,\mathrm{d}y &\leqslant 2^{nk}\int_{\mathbb{T}^n}\int_{B(x-z,2^{-k})} |u^y(x-z) - u^y(w)|^p \,\mathrm{d}y\mathrm{d}w \\ &= 2^{nk}\int_{\mathbb{T}^n}\int_{|h|\leqslant 2^{-k}} |u(t) - u(t-h)|^p \,\mathrm{d}h\mathrm{d}t, \quad \forall\, x,z \in \mathbb{T}^n.\end{aligned} \tag{2.49}$$

Using (2.47), we obtain

$$\begin{aligned}\sum_{k\geqslant 0} a_k^p &\leqslant \sum_{k\geqslant 0}\int_{\mathbb{T}^n}\int_{|h|\leqslant 2^{-k}} 2^{(n+sp)k}|u(t)-u(t-h)|^p\,\mathrm{d}h\mathrm{d}t \\ &= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\sum_{2^k\leqslant |h|^{-1}} 2^{(n+sp)k}|u(t)-u(t-h)|^p\,\mathrm{d}h\mathrm{d}t \\ &\leqslant c\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\frac{|u(t)-u(t-h)|^p}{|h|^{n+sp}}\,\mathrm{d}h\mathrm{d}t,\end{aligned}$$

with

$$c = c(n,s,p) := \sup_{|h|\leqslant 1} |h|^{n+sp}\sum_{2^k\leqslant|h|^{-1}} 2^{(n+sp)k} \leqslant 2.$$

Therefore, we have $\sum_{k\geqslant 0} a_k^p \leqslant 2\,|u|^p_{\mathrm{W}^{s,p}}$. □

Proof of Theorem 2.3 completed. — By the above lemma and Corollary 2.26 we have

$$\begin{aligned}L_1 &\lesssim \sum_{j\geqslant 0} 2^{(sp-1)j}\sum_{t=1}^{j+1} 2^t\left(\sum_{k=t-1}^{j} 2^{-sk}a_k\right)^p = \sum_{t\geqslant 1}\sum_{j\geqslant t-1} 2^{(sp-1)(j-t)}\left(\sum_{k=t-1}^{j} 2^{-s(k-t)}a_k\right)^p \\ &\lesssim \frac{1}{1-sp}\sum_{t\geqslant 1}\left(\sum_{k\geqslant t-1} 2^{-s(k-t)}a_k\right)^p \lesssim \frac{1}{s^p(1-sp)}\sum_{k\geqslant 0} a_k^p \lesssim \frac{1}{s^p(1-sp)}|u|^p_{\mathrm{W}^{s,p}}.\end{aligned}$$

By combining this with the estimate (2.46) of L_2, we find that

$$\int_{\mathbb{T}^n} |\varphi^y|^p_{\mathrm{W}^{s,p}} \,\mathrm{d}y \lesssim \frac{1}{s^p(1-sp)} |u|^p_{\mathrm{W}^{s,p}}. \qquad \square$$

Proof of Proposition 2.6. — As mentioned in the introduction, we rely on an argument devised for BV maps by Dávila and Ignat ([**23**]). Let $u \in \mathrm{W}^{s,1}(\mathbb{T}^n;\mathbb{S}^1)$. For every $\alpha \in \mathbb{S}^1$, we define $\varphi_\alpha := \theta_\alpha(u)$, where $\theta_\alpha(z)$ represents the unique argument of $z \in \mathbb{S}^1$ in the interval $(\alpha - 2\pi, \alpha]$. The functions φ_α are clearly measurable phases of u. We claim that there exists $\alpha \in \mathbb{S}^1$ such that $|\varphi_\alpha|_{\mathrm{W}^{s,1}} \leqslant 2|u|_{\mathrm{W}^{s,1}}$. For this purpose, we estimate the average of $|\varphi_\alpha|_{\mathrm{W}^{s,1}}$ over $\mathbb{S}^1$:

$$\begin{aligned}\fint_{\mathbb{S}^1} |\varphi_\alpha|_{\mathrm{W}^{s,1}} \,\mathrm{d}\alpha &= \fint_{\mathbb{S}^1} \left(\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|\varphi_\alpha(x) - \varphi_\alpha(y)|}{|x-y|^{n+s}} \,\mathrm{d}x\mathrm{d}y \right) \mathrm{d}\alpha \\ &= \frac{1}{2\pi} \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{1}{|x-y|^{n+s}} \left(\int_{\mathbb{S}^1} |\theta_\alpha(u(x)) - \theta_\alpha(u(y))| \,\mathrm{d}\alpha \right) \mathrm{d}x\mathrm{d}y.\end{aligned} \tag{2.50}$$

Applying Lemma 2.36 and using (2.50), we obtain

$$\fint_{\mathbb{S}^1} |\varphi_\alpha|_{\mathrm{W}^{s,1}} \,\mathrm{d}\alpha \leqslant 2|u|_{\mathrm{W}^{s,1}},$$

which proves the claim and completes the proof of the proposition. $\square$

2.7.3. Optimality when $sp < 1$. Proof of Theorem 2.5.

— The next result quantifies the asymptotic optimality of Theorem 2.3 in the special case where $n = 1$, $1 < p < \infty$ and $s = \dfrac{1-\varepsilon}{p}$, with $\varepsilon \to 0$. As we will see, the general case is an easy consequence of Proposition 2.8.

Proposition 2.8. — *For every* $\varepsilon \in \left(0, \frac{1}{2}\right)$, *there exists* $u_\varepsilon \in \mathrm{W}^{(1-\varepsilon)/p,p}(\mathbb{T};\mathbb{S}^1)$ *such that any phase* $\varphi \in \mathrm{W}^{(1-\varepsilon)/p,p}((0,1);\mathbb{R})$ *of* u_ε *satisfies*

$$|\varphi|_{\mathrm{W}^{(1-\varepsilon)/p,p}} \gtrsim \frac{1}{\varepsilon^{1/p}} |u_\varepsilon|_{\mathrm{W}^{(1-\varepsilon)/p,p}}.$$

The above proposition is a variant of [**11**, Theorem 2]. In turn, [**11**, Theorem 2] relies on a very involved result ([**11**, Theorem 1]) providing the asymptotic behavior of the best Sobolev constant in the embedding $\mathrm{W}^{1-\varepsilon,1}((0,1)) \hookrightarrow \mathrm{L}^{1/\varepsilon}((0,1))$. We present below a cousin argument, based on an inequality involving non-increasing rearrangements of functions, obtained by Garsia and Rodemich ([**28**]).

Proof of Proposition 2.8. — As in [**11**, Proof of Theorem 2], the key step consists in establishing the following estimate

$$|A|\,|{}^cA| \leqslant \left(C\varepsilon \int_A \int_{{}^cA} \frac{1}{|x-y|^{2-\varepsilon}} \,\mathrm{d}x\mathrm{d}y \right)^{1/\varepsilon}, \tag{2.51}$$

for every $\varepsilon \in \left(0, \dfrac{1}{2}\right)$ and every measurable set $A \subset (0,1)$. Here, ${}^cA$ is the complement of A, and C is an absolute constant.

Step 1. *Proof of* (2.51).

Recall that, if $f\colon (0,1) \to \mathbb{R}_+$ is a measurable function, then its non-increasing rearrangement $f^*\colon (0,1) \to \mathbb{R}_+$ is defined by

$$f^*(x) = \inf\left\{\lambda \in \mathbb{R} \,\middle|\, |\{t \in [0,1) \mid f(t) > \lambda\}| \leqslant x\right\}, \quad \forall\, x \in (0,1).$$

It is easy to see that, when $A \subset (0,1)$ is a measurable set, we have $(\mathbb{1}_A)^* = \mathbb{1}_{A^*}$, with $A^* = (0, |A|)$. Thus

$$(2.52) \quad \int_{A^*}\int_{{}^c(A^*)} \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{2-\varepsilon}} = \int_0^{|A|}\int_{|A|}^1 \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{2-\varepsilon}} = \frac{(1-|A|)^\varepsilon + |A|^\varepsilon - 1}{\varepsilon(1-\varepsilon)} = \frac{|A|^\varepsilon + |{}^cA|^\varepsilon - 1}{\varepsilon(1-\varepsilon)}.$$

On the other hand, we have

$$(2.53) \quad |A|^\varepsilon\, |{}^cA|^\varepsilon \lesssim (1-|A|)^\varepsilon + |A|^\varepsilon - 1$$

(see Lemma 2.37 in Subsection 2.7.7.4).

In view of (2.52) and (2.53), in order to establish (2.51) it suffices to prove that

$$\int_{A^*}\int_{{}^c(A^*)} \frac{1}{|x-y|^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y \leqslant \int_A\int_{{}^cA} \frac{1}{|x-y|^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y.$$

This is precisely the rearrangement inequality of Garsia and Rodemich ([**28**, Theorem I.1])

$$\int_0^1\int_0^1 \Psi\left(\frac{f^*(x)-f^*(y)}{p(x-y)}\right)\mathrm{d}x\mathrm{d}y \leqslant \int_0^1\int_0^1 \Psi\left(\frac{f(x)-f(y)}{p(x-y)}\right)\mathrm{d}x\mathrm{d}y,$$

applied with $f \coloneqq \mathbb{1}_A$, $p(t) \coloneqq |t|^{2-\varepsilon}$ and $\Psi(t) \coloneqq |t|$.

Step 2. *Proof of Proposition 2.8 completed.*

This part follows closely [**11**, Proof of Theorem 2], with some slight simplifications. We also detail some arguments which are only sketched in [**11**].

For $\delta \in \left(0, \dfrac{1}{2}\right)$, we define the phase

$$(2.54) \quad \varphi_\delta(x) \coloneqq \begin{cases} 0, & x < \dfrac{1}{2} \\ \dfrac{(2x-1)\pi}{\delta}, \text{ if} & \dfrac{1}{2} < x < \dfrac{1}{2} + \delta \\ 2\pi, & \dfrac{1}{2} + \delta < x \end{cases}.$$

We next choose $\delta = \delta(\varepsilon) \coloneqq \dfrac{1}{e^{1/\varepsilon}}$. For this choice of δ, the map $u_\varepsilon(x) \coloneqq \exp(\mathrm{i}\varphi_\delta(x))$, for $x \in (0,1)$, satisfies

$$(2.55) \quad |u|_{\mathrm{W}^{(1-\varepsilon)/p,p}((0,1))} \approx 1 \quad \text{when } \varepsilon \to 0$$

(see Lemma 2.38 in Subsection 2.7.7.4).

In order to prove Proposition 2.8, it suffices to show that any lifting φ of u_ε satisfies

$$|\varphi|_{W^{(1-\varepsilon)/p,p}} \gtrsim \frac{1}{\varepsilon^{1/p}}, \quad \text{for } \varepsilon \in \left(0, \frac{1}{2}\right).$$

Arguing by contradiction, we assume that, for every $\eta > 0$, there are some $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $\varphi \in W^{(1-\varepsilon)/p,p}((0,1);\mathbb{R})$ such that $u_\varepsilon \equiv \exp(i\varphi)$ and

$$(2.56) \quad |\varphi|^p_{W^{(1-\varepsilon)/p,p}} < \frac{\eta}{\varepsilon}.$$

We set $\psi \coloneqq \dfrac{\varphi - \varphi_\delta}{2\pi}$. Since both φ and φ_δ are liftings of u_ε, the function ψ takes its values into $\mathbb{Z}$. Straightforward calculations (see Lemma 2.39) show that

$$(2.57) \quad |\psi(x) - \psi(y)| \leqslant |\varphi(x) - \varphi(y)| \text{ if } x, y \in I_1 \coloneqq \left(0, \frac{1}{2} + \frac{2\delta}{3}\right), \text{ or if } x, y \in I_2 \coloneqq \left(\frac{1}{2} + \frac{\delta}{3}, 1\right).$$

We next invoke the following result, whose proof is postponed to Subsection 2.7.7.4.

Lemma 2.9. — *Let $I \subset \mathbb{R}$ be an interval and let $\psi\colon I \to \mathbb{Z}$ be any measurable function. Then there exists some $k \in \mathbb{Z}$ such that*

$$|\{x \in I \mid \psi(x) \neq k\}| \leqslant 4\left(C\varepsilon \int_I \int_I \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{2-\varepsilon}}\,dx dy\right)^{1/\varepsilon},$$

for all $\varepsilon \in \left(0, \frac{1}{2}\right)$, where C is the absolute constant in (2.51).

Step 2 continued.

Applying Lemma 2.9 with $I \coloneqq I_1$ and with $I \coloneqq I_2$ respectively, and using (2.57) together with (2.56), we obtain that there exist $m_1, m_2 \in \mathbb{Z}$ such that

$$(2.58) \quad |{}^c A_1| \leqslant 4\,(C\eta)^{1/\varepsilon} \quad \text{and} \quad |{}^c A_2| \leqslant 4\,(C\eta)^{1/\varepsilon},$$

where

$$A_1 \coloneqq \{x \in I_1 \mid \psi(x) = m_1\} \quad \text{and} \quad A_2 \coloneqq \{x \in I_2 \mid \psi(x) = m_2\}.$$

We now choose $\eta > 0$ such that $\eta < \dfrac{1}{\sqrt{24Ce}}$. With this choice of η, we have (using (2.58))

$$|A_1 \cap A_2| = |A_1 \cap A_2 \cap I_1 \cap I_2| \geqslant |I_1 \cap I_2| - |{}^c(A_1)| - |{}^c(A_2)| \geqslant \frac{\delta(\varepsilon)}{3} - 8\,(C\eta)^{1/\varepsilon} > 0,$$

and thus we must have $m_1 = m_2$. We may further assume that $m_1 = m_2 = 0$.

Consider the following sets:

$$B_1 \coloneqq \left\{x \in \left(0, \frac{1}{2}\right) \,\middle|\, \psi(x) \neq 0\right\} \subset \left(0, \frac{1}{2}\right)$$

and

$$B_2 \coloneqq \left\{x \in \left(\frac{1}{2} + \delta, 1\right) \,\middle|\, \psi(x) \neq 0\right\} \subset \left(\frac{1}{2} + \delta, 1\right).$$

We clearly have

$$\varphi = \varphi_\delta \quad \text{on} \quad {}^c B_1 \qquad \text{and} \qquad \varphi = \varphi_\delta \quad \text{on} \quad {}^c B_2$$

and, in addition, by (2.58) we also have

$$(2.59) \quad |B_1| \leqslant \frac{\delta}{6} \quad \text{and} \quad |B_2| \leqslant \frac{\delta}{6}.$$

By the definition of φ_δ, we then find

$$\begin{aligned}|\varphi|^p_{W^{(1-\varepsilon)/p,p}} &\geqslant \int_{^cB_1}\int_{^cB_2}\frac{|\varphi(x)-\varphi(y)|^p}{(x-y)^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y = \int_{^cB_1}\int_{^cB_2}\frac{|\varphi_\delta(x)-\varphi_\delta(y)|^p}{(x-y)^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y\\ &\geqslant (2\pi)^p\int_{^cB_1}\int_{^cB_2}\frac{1}{(x-y)^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

It is easy to see that the latter quantity does not increase if the sets B_1 and B_2 are replaced respectively by the intervals

$$\widetilde{B}_1 \coloneqq \left(\frac{1}{2}-|B_1|,\frac{1}{2}\right) \quad \text{and} \quad \widetilde{B}_2 \coloneqq \left(\frac{1}{2}+\delta,\frac{1}{2}+\delta+|B_2|\right);$$

see Lemma 2.40. Hence, using (2.59) and the fact that $\delta = \dfrac{1}{e^{1/\varepsilon}}$, we obtain

$$(2.60)\quad |\varphi|^p_{W^{(1-\varepsilon)/p,p}} \geqslant (2\pi)^p\int_0^{1/2-\delta/6}\int_{1/2+7\delta/6}^1\frac{1}{(x-y)^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y = \frac{(2\pi)^p}{\varepsilon(1-\varepsilon)}\left(1-\frac{1}{e}+o(1)\right),$$

when $\varepsilon\to 0$. For an appropriate choice of η, (2.60) contradicts (2.56). □

Proof of Theorem 2.5. — The optimality of the estimate (2.30) in Theorem 2.3 means that for every $0<s<1$ with $1-sp\ll 1$ there exists a map $u\in W^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ such that any lifting $\varphi\in W^{s,p}((0,1)^n;\mathbb{R})$ of u satisfies

$$(2.61)\quad |\varphi|^p_{W^{s,p}} \gtrsim \frac{1}{1-sp}|u|^p_{W^{s,p}}.$$

This is true in dimension $n=1$ by the above Proposition 2.8. In order to prove (2.61) in arbitrary dimension, we use a dimensional reduction argument. More specifically, for every $s\in\left(\dfrac{1}{2p},\dfrac{1}{p}\right)$ we define

$$u(x)=u(x_1,x_2,\dots,x_n) \eqqcolon u(x_1,x') \coloneqq u_s(x_1),\quad \forall\, x\in\mathbb{T}^n.$$

Here, u_s is a map in $W^{s,p}(\mathbb{T};\mathbb{S}^1)$ that satisfies the property that for any lifting $\varphi\in W^{s,p}((0,1);\mathbb{R})$ of u_s we have

$$(2.62)\quad |\varphi|^p_{W^{s,p}((0,1))} \gtrsim \frac{1}{1-sp}|u_s|^p_{W^{s,p}(\mathbb{T})}.$$

Note that the existence of u_s follows from Proposition 2.8.

Consider an arbitrary lifting $\psi\in W^{s,p}((0,1)^n;\mathbb{R})$ of u. Clearly, for almost every $x' \coloneqq (x_2,\dots,x_n)\in(0,1)^{n-1}$, the map $x_1\mapsto\psi(x_1,x') \eqqcolon \varphi_{x'}(x_1)$ is a lifting of u_s, and thus satisfies the estimate (2.62). By combining this fact with Corollary 2.43, we find

$$(2.63)\quad |\psi|^p_{W^{s,p}((0,1)^n)} \approx |\varphi_{x'}|^p_{W^{s,p}((0,1))} \gtrsim \frac{1}{1-sp}|u_s|^p_{W^{s,p}(\mathbb{T})}.$$

On the other hand, we have, again by Corollary 2.43, that $|u|_{W^{s,p}(\mathbb{T}^n)}\approx|u_s|_{W^{s,p}(\mathbb{T})}$, which, together with (2.63), leads to

$$|\psi|^p_{W^{s,p}((0,1)^n)} \gtrsim \frac{1}{1-sp}|u|^p_{W^{s,p}(\mathbb{T}^n)}.$$ □

2.7.4. Optimal estimates when $sp \geqslant 1$. — As we will see below, when $sp \geqslant 1$, two quantitatively different types of estimates occur: linear estimates of the form $|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}}$, and superlinear estimates of the form

$$|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}} + |u|^{\alpha}_{W^{s,p}}, \tag{2.64}$$

with $\alpha > 1$.

The linear regime corresponds to the case where $s \geqslant 1$. When $s = 1$, we actually have the identity $|\varphi|_{W^{1,p}} = |u|_{W^{1,p}}$, and optimality is irrelevant. When $s > 1$, several natural semi-norms $|\cdot|_{W^{s,p}}$ can be considered (where a natural semi-norm is a semi-norm modulo constant functions and equivalent to the standard norm on the quotient space $W^{s,p}/\mathbb{C}$), and optimal estimates do depend on the choice of such a semi-norm. Therefore, we restrict ourselves to a more modest task, which consists in proving that optimal estimates are indeed linear.

When $s < 1$, we will obtain superlinear estimates of type (2.64). In this case, we focus on the optimality of the exponent α (when $|u|_{W^{s,p}}$ is large) and of the linear term $|u|_{W^{s,p}}$ (when $|u|_{W^{s,p}}$ is small).

Theorem 2.10. — *Let $s \geqslant 1$, $1 \leqslant p < \infty$ be such that $sp \geqslant 2$. Let $u \in W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$ and let $\varphi \in W^{s,p}(\mathbb{T}^n; \mathbb{R})$ be a lifting of u. Then*

$$|\varphi|_{W^{s,p}} \leqslant C(s,p)|u|_{W^{s,p}}. \tag{2.65}$$

Moreover, the above estimate is optimal in the sense that

$$\limsup_{|\varphi|_{W^{s,p}} \to 0} \frac{|\varphi|_{W^{s,p}}}{|u|_{W^{s,p}}} > 0, \quad \textit{and} \quad \limsup_{|u|_{W^{s,p}} \to \infty} \frac{|\varphi|_{W^{s,p}}}{|u|_{W^{s,p}}} > 0.$$

Proof. — Since the estimate (2.65) does not depend on the choice of the semi-norm, we can work for convenience with the semi-norm $|f|_{W^{s,p}} \coloneqq \|\nabla f\|_{W^{s-1,p}}$.

Step 1. *Proof of* (2.65).

Since $s \geqslant 1$, we may differentiate once the equality $u = e^{i\varphi}$ and find that $\nabla\varphi = u \wedge \nabla u$ $\big($recall that $(u_1 + iu_2) \wedge \nabla(v_1 + iv_2) = u_1\nabla v_2 - u_2\nabla v_1\big)$. Thus we have to establish the estimate $\|u \wedge \nabla u\|_{W^{s-1,p}} \lesssim |u|_{W^{s,p}}$. We first extend u to a map in $\mathbb{R}^n$ using a standard extension operator $P\colon W^{s,p}(\mathbb{T}^n) \to W^{s,p}(\mathbb{R}^n)$. This goes as follows. We first define $v \coloneqq P\left(u - \int_{\mathbb{T}^n} u\right)$, which belongs to $W^{s,p}(\mathbb{R}^n)$ and then $w \coloneqq v + \int_{\mathbb{T}^n} u$ is an extension of u. We next note that

$$\begin{aligned}
\|u \wedge \nabla u\|_{W^{s-1,p}(\mathbb{T}^n)} \leqslant \|w \wedge \nabla w\|_{W^{s-1,p}(\mathbb{R}^n)} &= \left\|\left(v + \int_{\mathbb{T}^n} u\right) \wedge \nabla v\right\|_{W^{s-1,p}(\mathbb{R}^n)} \\
&\lesssim \|v \wedge \nabla v\|_{W^{s-1,p}(\mathbb{R}^n)} + \|\nabla v\|_{W^{s-1,p}(\mathbb{R}^n)} \\
&= \|v \wedge \nabla v\|_{W^{s-1,p}(\mathbb{R}^n)} + |v|_{W^{s,p}(\mathbb{R}^n)}.
\end{aligned}$$

In the last inequality we used the fact that $\left|\fint u\right| \leqslant 1$. Therefore, by Lemma 2.45, and by the fact that $|v| \leqslant 2$, we obtain

$$\|u \wedge \nabla u\|_{\mathrm{W}^{s-1,p}(\mathbb{T}^n)} \lesssim \|v\|_{\mathrm{W}^{s,p}(\mathbb{R}^n)} = \left\|P\left(u - \int_{\mathbb{T}^n} u\right)\right\|_{\mathrm{W}^{s,p}(\mathbb{R}^n)} \lesssim \left\|u - \int_{\mathbb{T}^n} u\right\|_{\mathrm{W}^{s,p}(\mathbb{T}^n)}$$
$$\lesssim |u|_{\mathrm{W}^{s-1,p}(\mathbb{T}^n)}$$

(the last inequality following from Poincaré's inequality).

Step 2. *Optimality in dimension $n = 1$.*

The optimality of (2.65) needs to be checked for $|\varphi|_{\mathrm{W}^{s,p}} \to 0$ and for $|u|_{\mathrm{W}^{s,p}} \to \infty$, that is, we need to show that:

1. There exists $(\varphi_j)_{j\geqslant 1}$ in $\mathrm{W}^{s,p}(\mathbb{T};\mathbb{R})$ such that $|\varphi_j|_{\mathrm{W}^{s,p}} \to 0$ and $|u_j|_{\mathrm{W}^{s,p}} \lesssim |\varphi_j|_{\mathrm{W}^{s,p}}$, where $u_j \coloneqq e^{\mathrm{i}\varphi_j}$.
2. There exists $(u_j)_{j\geqslant 1}$ in $\mathrm{W}^{s,p}(\mathbb{T};\mathbb{S}^1)$ such that $|u_j|_{\mathrm{W}^{s,p}} \to \infty$ and $|u_j|_{\mathrm{W}^{s,p}} \lesssim |\varphi_j|_{\mathrm{W}^{s,p}}$, where φ_j is the (unique modulo 2π) lifting of u_j.

For the optimality "at zero", we let $f \in C^\infty_c((0,1);\mathbb{R})$, and define $\varphi_j \coloneqq \dfrac{1}{j} f$, and $u_j \coloneqq e^{\mathrm{i}\varphi_j}$. Clearly, we have

$$\text{(2.66)} \qquad |\varphi_j|_{\mathrm{W}^{r,p}} = \frac{|f|_{\mathrm{W}^{r,p}}}{j} \approx \frac{1}{j} \to 0 \quad \text{as} \quad j \to \infty, \quad \forall\, r > 0.$$

Using (2.66) and a straightforward induction, it is easy to see that, when $k \geqslant 1$ is an integer, we may write

$$\text{(2.67)} \qquad \mathrm{D}^k u_j = g_{k,j} u_j, \text{ for some } g_{k,j} \in C^\infty_c((0,1);\mathbb{C}) \text{ such that } |g_{k,j}|_{\mathrm{W}^{r,p}} \lesssim \frac{1}{j}, \quad \forall\, r > 0.$$

We now establish item 1 for the above choice of φ_j and u_j. Assume first that s is an integer. Then by (2.67) we have

$$\text{(2.68)} \qquad |u_j|_{\mathrm{W}^{s,p}} = \|\nabla u_j\|_{\mathrm{W}^{s-1,p}} = \sum_{1\leqslant k\leqslant s} \left\|\mathrm{D}^k u_j\right\|_{\mathrm{L}^p} \lesssim \frac{1}{j}.$$

By (2.66) and (2.68), we find that $|u_j|_{\mathrm{W}^{s,p}} \lesssim |\varphi_j|_{\mathrm{W}^{s,p}}$.

Assume next that s is not an integer and set $\sigma \coloneqq s - [s] \in (0,1)$. By (2.68), we have

$$\text{(2.69)} \qquad \left\|\mathrm{D}^k u_j\right\|_{\mathrm{L}^p} \lesssim \frac{1}{j}, \quad \forall\, 1 \leqslant k \leqslant [s].$$

As a consequence of (2.69) with $k = 1$, we also have

$$\text{(2.70)} \qquad |u_j|_{\mathrm{W}^{r,p}} \lesssim \frac{1}{j}, \quad \forall\, r \in (0,1).$$

In order to conclude, it suffices to establish the estimate $\left|\mathrm{D}^{[s]} u_j\right|_{\mathrm{W}^{\sigma,p}} \lesssim \dfrac{1}{j}$. This estimate is an immediate consequence of (2.67), of (2.70) and of the inequality

$$\text{(2.71)} \qquad \left|\mathrm{D}^{[s]} u_j\right|^p_{\mathrm{W}^{\sigma,p}} \lesssim |u_j|^p_{\mathrm{W}^{\sigma,p}} + \left|g_{[s],j}\right|^p_{\mathrm{W}^{s,p}} \lesssim |u_j|^p_{\mathrm{W}^{\sigma,p}} + \frac{1}{j^p}.$$

In turn, (2.71) follows from

$$\begin{aligned}\left|g_{[s],j}(x)u_j(x) - g_{[s],j}(y)u_j(y)\right| &\leqslant \left|g_{[s],j}(x)\right| |u_j(x) - u_j(y)| + \left|g_{[s],j}(x) - g_{[s],j}(y)\right| \\ &\lesssim |u_j(x) - u_j(y)| + \left|g_{[s],j}(x) - g_{[s],j}(y)\right|.\end{aligned}$$

In order to prove the optimality of (2.65) "at infinity" we take φ_j to be a sum of j copies of a properly scaled C_c^∞ function. More precisely, we fix $f \in \mathrm{C}_c^\infty((0,1);\mathbb{R})$ and we define the functions $\varphi_j := \sum_{k=0}^{j-1} f(xj-k)$, $\forall\, j \geqslant 1$, whose semi-norms can be estimated by

$$(2.72)\qquad |\varphi_j|_{\mathrm{W}^{s,p}} \approx j^s$$

(see Lemma 2.44 and recall that $A \approx B$ stands for $B \lesssim A \lesssim B$). Next, we take $g := e^{\mathrm{i}f} - 1$, which belongs to $\mathrm{C}_c^\infty((0,1);\mathbb{C})$, and

$$u_j := \sum_{k=0}^{j-1} g(xj-k) + 1, \quad \forall\, j \geqslant 1.$$

Since $u_j(x) = \sum_{k=0}^{j-1}(e^{\mathrm{i}f(xj-k)} - 1) + 1 = \prod_{k=0}^{j-1} e^{\mathrm{i}f(xj-k)} = e^{\mathrm{i}\varphi_j(x)}$, the function φ_j is "the" lifting of u_j. On the other hand, by Lemma 2.44 we have

$$(2.73)\qquad |u_j|_{\mathrm{W}^{s,p}} \approx j^s.$$

By (2.72) and (2.73), we have $|u_j|_{\mathrm{W}^{s,p}} \approx |\varphi_j|_{\mathrm{W}^{s,p}} \to \infty$ when $j \to \infty$, which proves item 2 when $n = 1$.

Step 3. *Optimality in higher dimension.*

Let φ_j, u_j be as in **Step 2**. As in the proof of Proposition 2.8, we let:

$$(2.74)\qquad \psi_j(x_1, x') := \varphi_j(x_1), \quad v_j(x_1, x') := u_j(x_1) = e^{\mathrm{i}\varphi_j(x_1)}, \quad \forall\, x_1 \in \mathbb{T},\ x' \in \mathbb{T}^{n-1}.$$

Then $v_j = e^{\mathrm{i}\psi_j}$ and, by Corollary 2.43, we have the equivalence of norms $|\psi_j|_{\mathrm{W}^{s,p}} \approx |\varphi_j|_{\mathrm{W}^{s,p}}$ and $|v_j|_{\mathrm{W}^{s,p}} \approx |u_j|_{\mathrm{W}^{s,p}}$. Therefore, since φ_j and u_j were chosen such that $|u_j|_{\mathrm{W}^{s,p}} \lesssim |\varphi_j|_{\mathrm{W}^{s,p}}$, we also have $|v_j|_{\mathrm{W}^{s,p}} \lesssim |\psi_j|_{\mathrm{W}^{s,p}}$. □

We next turn to the case where $0 < s < 1$. In view of [**10**] (see also Subsection 2.7.1), when $0 < s < 1$, $\mathrm{W}^{s,p}$ has the lifting property if and only if $sp \geqslant n$. We start by presenting an exceptional case, already observed in [**10**], where there is no possible estimate of φ in terms of u. More specifically, we have the following:

Proposition 2.11. — *([**10**]) Let $1 < p < \infty$. Then there is no estimate of the form $|\varphi|_{\mathrm{W}^{1/p,p}} \leqslant F\left(|u|_{\mathrm{W}^{1/p,p}}\right)$.*

Let us briefly recall the argument in [**10**]. Let φ_δ be as in (2.54) and set $u_\delta := e^{\mathrm{i}\varphi_\delta}$. Then it is easily checked that $|u_\delta|_{\mathrm{W}^{1/p,p}} \lesssim 1$ and $|\varphi|_{\mathrm{W}^{1/p,p}} \to \infty$ as $\delta \to 0$. Since φ_δ is the unique phase (mod 2π) of u_δ, we obtain the non-existence of an estimate of the form $|\varphi|_{\mathrm{W}^{1/p,p}} \leqslant F(|u|_{\mathrm{W}^{1/p,p}})$.

As we will see below, this is the only exceptional case. In the remaining cases, we will establish several positive results. We start by recalling an elementary estimate, due to Merlet [**38**, Theorem 1.1], and whose proof is postponed.

Theorem 2.12 ([**38**]). — *Let $n = 1$. Assume that $0 < s < 1$, $1 < p < \infty$ and $sp > 1$. Let $u \in \mathrm{W}^{s,p}(\mathbb{T};\mathbb{S}^1)$ and let $\varphi \in \mathrm{W}^{s,p}(\mathbb{T};\mathbb{R})$ be a lifting of u. Then*

$$|\varphi|_{\mathrm{W}^{s,p}} \lesssim |u|_{\mathrm{W}^{s,p}} + |u|^{1/s}_{\mathrm{W}^{s,p}}. \tag{2.75}$$

In higher dimensions, we obtain the same result as the one in Theorem 2.12, but the corresponding proof is much more involved.

Theorem 2.13. — *Let $n \geq 2$. Assume that $0 < s < 1$, $1 < p < \infty$ and $sp \geq n$. Let $u \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ and let $\varphi \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{R})$ be a lifting of u. Then*

$$|\varphi|_{\mathrm{W}^{s,p}} \lesssim |u|_{\mathrm{W}^{s,p}} + |u|^{1/s}_{\mathrm{W}^{s,p}}. \tag{2.76}$$

We start with the

Proof of Theorem 2.13. — Estimate (2.76) will be obtained via the factorization method presented in [**40**]. More precisely, the arguments in [**40**], that we will detail below, lead to the existence of some $\varphi_1 \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{R})$ such that

$$|\varphi_1|_{\mathrm{W}^{s,p}} \lesssim |u|_{\mathrm{W}^{s,p}} \quad \text{and} \quad \left\|\nabla\left(ue^{-\mathrm{i}\varphi_1}\right)\right\|_{\mathrm{L}^{sp}} \lesssim |u|^{1/s}_{\mathrm{W}^{s,p}}.$$

The construction of the map φ_1 goes as follows. First, by suitably extending u (as in **Step 1** in the proof of Theorem 2.10) we may identify u with a map in $\mathbb{R}^n$, still denoted u, with the following properties:

1. $|u|_{\mathrm{W}^{s,p}(\mathbb{R}^n)} \lesssim |u|_{\mathrm{W}^{s,p}(\mathbb{T}^n)}$.
2. $|u| \leqslant 2$.
3. u is $\mathbb{S}^1$-valued in $(-3,4)^n$.
4. u is constant outside $(-4,5)^n$.

We next consider a mollifier $\rho \in C^\infty_c(\mathbb{R}^n)$ satisfying: $\rho \geqslant 0$, $\displaystyle\int_{\mathbb{R}^n} \rho = 1$ and $\operatorname{supp}\rho \subset B(0,2) \setminus \overline{B(0,1)}$. We then let

$$w(x,\varepsilon) \coloneqq u * \rho_\varepsilon(x), \quad \forall\, x \in \mathbb{R}^n,\ \forall\, \varepsilon > 0, \tag{2.77}$$

and define

$$\varphi_1(x) \coloneqq -\int_0^\infty \Pi \circ w(x,\varepsilon) \wedge \frac{\partial}{\partial\varepsilon}\Pi \circ w(x,\varepsilon)\,\mathrm{d}\varepsilon. \tag{2.78}$$

Here,

$$\Pi \in C^\infty(\mathbb{R}^2;\mathbb{R}^2) \quad \text{and} \quad \Pi(z) = \frac{z}{|z|} \text{ when } |z| \geqslant \frac{1}{2}. \tag{2.79}$$

We now explain the motivation behind this construction. Intuitively, φ_1 encodes the small amplitude oscillations of u, while the remainder $u\,e^{-\mathrm{i}\varphi_1}$ encodes the large amplitude oscillations $\left(\text{as those contained in the topological singularities of type } \frac{z}{|z|}\right)$. The reason is the following. Assume that u has only small amplitude oscillations, say around the value 1. Then the extension w of u is still close to 1, and thus the restriction of $\Pi \circ w$ to $\mathbb{T}^n \times (0,\infty)$ is a smooth $\mathbb{S}^1$-valued extension of u. It follows that $\Pi \circ w$ has a smooth phase ψ. By differentiating the identity $\Pi \circ w \equiv e^{\mathrm{i}\psi}$, we find that

$$\frac{\partial}{\partial\varepsilon}\psi(x,\varepsilon) = \Pi \circ w(x,\varepsilon) \wedge \frac{\partial}{\partial\varepsilon}(\Pi \circ w)(x,\varepsilon), \quad \forall\, x \in \mathbb{T}^n,\ \forall\, \varepsilon > 0. \tag{2.80}$$

Assuming in addition that $\Pi\circ w$ converges sufficiently fast to 1 as $\varepsilon\to\infty$, we may integrate (2.80) and find that

$$u(x)=\lim_{\varepsilon\to 0} w(x,\varepsilon)=e^{\mathrm{i}\varphi_1(x)} \text{ for a.e. } x, \text{ with } \varphi_1 \text{ given by (2.78).}$$

Therefore, φ_1 gives (under some reasonable assumptions) a phase of u provided u has small amplitude oscillations. In general, u need not have small amplitude oscillations, and the remainder $u\,e^{-\mathrm{i}\varphi_1}$ measures what is left, i.e., the large amplitude oscillations.

We now turn to the implications of this construction for the proof of Theorem 2.13. The next two results are from [**40**].

Lemma 2.14. — *Let $n\geqslant 1$, $0<s<1$ and $1\leqslant p<\infty$. Let $u:\mathbb{R}^n\to\mathbb{C}$ satisfy items 1, 2 and 4 above. Let φ_1 be as in (2.78). Then:*

1. *The function φ_1 is well-defined a.e. on $\mathbb{T}^n$, in the sense that the integral in (2.78) is absolutely convergent for a.e. $x\in\mathbb{T}^n$.*
2. *$\varphi_1\in \mathrm{W}^{s,p}(\mathbb{T}^n)$ and*

$$|\varphi_1|_{\mathrm{W}^{s,p}}\lesssim |u|_{\mathrm{W}^{s,p}}. \tag{2.81}$$

Lemma 2.15. — *Let $n\geqslant 1$, $s>0$ and $1\leqslant p<\infty$ be such that $sp\geqslant 1$. Let $u:\mathbb{R}^n\to\mathbb{C}$ satisfy properties 1–4 above. Let φ_1 be as in (2.78).*

Then $ue^{-\mathrm{i}\varphi_1}\in \mathrm{W}^{1,sp}(\mathbb{T}^n;\mathbb{S}^1)$ and

$$\left\|\nabla\left(ue^{-\mathrm{i}\varphi_1}\right)\right\|_{\mathrm{L}^{sp}(\mathbb{T}^n)}\lesssim |u|^{1/s}_{\mathrm{W}^{s,p}(\mathbb{T}^n)}. \tag{2.82}$$

Proof of Theorem 2.13 completed. Let φ_1 be as in (2.78). By Lemma 2.15, the map $ue^{-\mathrm{i}\varphi_1}$ belongs to the space $\mathrm{W}^{1,sp}(\mathbb{T}^n;\mathbb{S}^1)$. Since $sp\geqslant 2$, by Theorem 2.1 we may write $ue^{-\mathrm{i}\varphi_1}=e^{\mathrm{i}\varphi_2}$ with $\varphi_2\in \mathrm{W}^{1,sp}(\mathbb{T}^n)$. Since $sp\geqslant n$, we have $\mathrm{W}^{1,sp}(\mathbb{T}^n)\hookrightarrow \mathrm{W}^{s,p}(\mathbb{T}^n)$, and thus $u=e^{\mathrm{i}\varphi}$ with $\varphi:=\varphi_1+\varphi_2\in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{R})$. Since $sp\geqslant 1$, φ is the unique (mod 2π) phase of u in $\mathrm{W}^{s,p}$ (by [**10**, Theorem B.1]). Moreover, using (2.81) and (2.82), we can estimate $|\varphi|_{\mathrm{W}^{s,p}}$ as follows.

$$\begin{aligned}|\varphi|_{\mathrm{W}^{s,p}}&\lesssim|\varphi_1|_{\mathrm{W}^{s,p}}+|\varphi_2|_{\mathrm{W}^{s,p}}\lesssim|u|_{\mathrm{W}^{s,p}}+|\varphi_2|_{\mathrm{W}^{1,sp}}=|u|_{\mathrm{W}^{s,p}}+\|\nabla\varphi_2\|_{\mathrm{L}^{sp}}\\&=|u|_{\mathrm{W}^{s,p}}+\left\|\nabla\left(ue^{-\mathrm{i}\varphi_1}\right)\right\|_{\mathrm{L}^{sp}}\lesssim|u|_{\mathrm{W}^{s,p}}+|u|^{1/s}_{\mathrm{W}^{s,p}}.\end{aligned} \tag{2.83}$$

□

We now turn to Theorem 2.12 and present three different proofs, with different flavors. The first one is a variant of the proof of Theorem 2.13. The second one simplifies Merlet's original argument. The third one is non constructive (unlike the proof of Theorem 2.13) and is inspired by an argument in Nguyen [**48**].

First proof of Theorem 2.12. — The following argument is similar to the one in Theorem 2.13. We consider the phase φ_1 defined therein. This time, we have $ue^{-\mathrm{i}\varphi_1}$ in $\mathrm{W}^{1,sp}(\mathbb{T};\mathbb{S}^1)$ with $sp\geqslant 1$. Since (by Theorem 2.1) in dimension $n=1$ all the Sobolev spaces do have the lifting property, we can write $ue^{-\mathrm{i}\varphi_1}=e^{\mathrm{i}\varphi_2}$ with $\varphi_2\in \mathrm{W}^{1,sp}(\mathbb{T})$. Since $sp>1$, we have $\mathrm{W}^{1,sp}(\mathbb{T})\hookrightarrow \mathrm{W}^{s,p}(\mathbb{T})$, and thus $u=e^{\mathrm{i}\varphi}$ with $\varphi:=\varphi_1+\varphi_2\in \mathrm{W}^{s,p}(\mathbb{T};\mathbb{R})$. We now obtain the estimate (2.75) following the argument leading to (2.83). □

Second proof of Theorem 2.12. — The starting point is the estimate (2.84) below, due to Merlet [**38**]:

$$|\varphi(x)-\varphi(y)|^p \lesssim |u(x)-u(y)|^p + (y-x)^{p-1/s}|u|^{p/s}_{\mathrm{W}^{s,p}((x,y))}, \quad 0 \leqslant x < y \leqslant 1. \tag{2.84}$$

(For a simplification of Merlet's original argument leading to (2.84), see the proof of Lemma 2.49.)

Dividing the inequality (2.84) by $(y-x)^{1+sp}$ and then integrating in x and y, we find that

$$|\varphi|^p_{\mathrm{W}^{s,p}(\mathbb{T})} \lesssim |u|^p_{\mathrm{W}^{s,p}(\mathbb{T})} + \int_0^1\int_0^y \frac{|u|^{p/s}_{\mathrm{W}^{s,p}((x,y))}}{(y-x)^\alpha}\,\mathrm{d}x\mathrm{d}y, \tag{2.85}$$

with $\alpha \coloneqq 1+sp-p+\dfrac{1}{s}$. Next we note that, since $s<1$, we have $\dfrac{p}{s}>p$ and therefore

$$|u|^{p/s}_{\mathrm{W}^{s,p}((x,y))} \leqslant |u|^{p/s-p}_{\mathrm{W}^{s,p}(\mathbb{T})}\,|u|^p_{\mathrm{W}^{s,p}((x,y))}. \tag{2.86}$$

On the other hand, since $s<1$ and $sp>1$, we have $\alpha<2$. We obtain

$$\begin{aligned}\int_0^1\int_0^y \frac{|u|^p_{\mathrm{W}^{s,p}((x,y))}}{(y-x)^\alpha}\,\mathrm{d}x\mathrm{d}y &\approx \int_0^1\int_0^y \frac{1}{(y-x)^\alpha}\int_x^y\int_x^z \frac{|u(z)-u(t)|^p}{(z-t)^{1+sp}}\,\mathrm{d}t\mathrm{d}z\mathrm{d}x\mathrm{d}y\\ &= \int_0^1\int_0^z \frac{|u(z)-u(t)|^p}{(z-t)^{1+sp}}\int_z^1\int_0^t \frac{1}{(y-x)^\alpha}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}z\\ &\leqslant \int_0^1\int_0^z \frac{|u(z)-u(t)|^p}{(z-t)^{1+sp}}\int_t^{t+1}\int_{t-1}^t \frac{1}{(y-x)^\alpha}\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}z\\ &\leqslant C\,|u|^p_{\mathrm{W}^{s,p}(\mathbb{T})}.\end{aligned}$$

Here, $C \coloneqq \displaystyle\int_{-1}^0\int_0^1 \frac{1}{(y-x)^\alpha}\,\mathrm{d}x\mathrm{d}y < \infty$ (since $\alpha<2$). The above inequality together with (2.85) and (2.86) implies $|\varphi|^p_{\mathrm{W}^{s,p}} \lesssim |u|^p_{\mathrm{W}^{s,p}} + |u|^{p/s}_{\mathrm{W}^{s,p}}$. Thus (2.76) holds. □

Third proof of Theorem 2.12. —

Step 1. *Proof of* (2.76) *when u is smooth and has a smooth periodic phase.*

Suppose that u belongs to $\mathrm{C}^\infty(\mathbb{T};\mathbb{S}^1)$ and that we may write $u=e^{\mathrm{i}\varphi}$, with $\varphi\in\mathrm{C}^\infty(\mathbb{T};\mathbb{R})$ (this is equivalent to $\deg(u;\mathbb{T})=0$). In this case, we will prove the existence of two linear maps, T_1 and T_2, such that for every $\zeta\in\mathrm{C}^\infty(\mathbb{T};\mathbb{R})$ we have

1. $\displaystyle\int_{\mathbb{T}}\varphi'(x)\zeta(x)\,\mathrm{d}x = T_1(\zeta)+T_2(\zeta)$.
2. $T_1(1)=T_2(1)=0$.
3. $|T_1(\zeta)| \lesssim \|\zeta\|_{\mathrm{L}^{(sp)'}}|u|^{1/s}_{\mathrm{W}^{s,p}}$.
4. $|T_2(\zeta)| \lesssim |\zeta|_{\mathrm{W}^{1-s,p'}}|u|_{\mathrm{W}^{s,p}}$.

Assume for the moment that items 1 to 4 are proved. Using the dualities $\left(\mathrm{L}^{(sp)'}\right)^*=\mathrm{L}^{sp}$, respectively $\left(\mathrm{W}^{1-s,p'}\right)^*=\mathrm{W}^{s-1,p}$, we find that there exist some $\psi_1\in\mathrm{L}^{sp}$ and $\psi_2\in\mathrm{W}^{s-1,p}$ such that

a) $\varphi'=\psi_1+\psi_2$ in the distributional sense.

b) $\displaystyle\int_{\mathbb{T}}\psi_1=0$ and $\psi_2(1)=0$.

c) $\|\psi_1\|_{\mathrm{L}^{sp}} \lesssim |u|_{\mathrm{W}^{s,p}}^{1/s}$.

d) $\|\psi_2\|_{\mathrm{W}^{s-1,p}} \lesssim |u|_{\mathrm{W}^{s,p}}$.

Item d) requires a proof. In view of item 4, of the Poincaré inequality $\left\|u - \fint u\right\|_{\mathrm{W}^{\sigma,q}} \lesssim |u|_{\mathrm{W}^{\sigma,q}}$ (with $0 < \sigma < 1$ and $1 \leqslant q \leqslant \infty$) and of the fact that $T_2(1) = 0$, we find that $|T_2(\zeta)| = \left|T_2\left(\zeta - \fint \zeta\right)\right| \lesssim |\zeta|_{\mathrm{W}^{1-s,p'}}|u|_{\mathrm{W}^{s,p}} \lesssim \|\zeta\|_{\mathrm{W}^{1-s,p'}}|u|_{\mathrm{W}^{s,p}}$. This leads to item d).

By estimates c) and d) and by property b), we may find some $\varphi_1 \in \mathrm{W}^{1,sp}$ and $\varphi_2 \in \mathrm{W}^{s,p}$ such that $\varphi_1' = \psi_1$ and $\varphi_2' = \psi_2$. In addition, we note the estimates $|\varphi_1|_{\mathrm{W}^{1,sp}} \lesssim |u|_{\mathrm{W}^{s,p}}^{1/s}$ and $|\varphi_2|_{\mathrm{W}^{s,p}} \lesssim |u|_{\mathrm{W}^{s,p}}$. By construction, we have $\varphi = \varphi_1 + \varphi_2$ (up to an additive constant). Using the Sobolev embedding $\mathrm{W}^{1,sp}(\mathbb{T}) \hookrightarrow \mathrm{W}^{s,p}(\mathbb{T})$, we obtain

$$|\varphi|_{\mathrm{W}^{s,p}} \leqslant |\varphi_2|_{\mathrm{W}^{s,p}} + |\varphi_1|_{\mathrm{W}^{s,p}} \lesssim |\varphi_2|_{\mathrm{W}^{s,p}} + |\varphi_1|_{\mathrm{W}^{1,sp}} \lesssim |u|_{\mathrm{W}^{s,p}} + |u|_{\mathrm{W}^{s,p}}^{1/s},$$

which is the desired conclusion.

So let us construct T_1 and T_2 satisfying items 1 – 4. We identify $\mathbb{T}$ with the boundary $\mathbb{S}^1$ of the unit disc $\mathbb{D}$ and we identify the derivative on $\mathbb{T}$ with the tangential derivative on $\mathbb{S}^1$. Let ξ be the harmonic extension to $\mathbb{D}$ of ζ, and let $\widetilde{u} = \widetilde{u}_1 + \mathrm{i}\widetilde{u}_2$ be a smooth extension of $u = u_1 + \mathrm{i}u_2$ to $\mathbb{D}$ (the choice of $\widetilde{u}$ will be specified later). Noting the fact that the Jacobian determinant $\mathrm{Jac}(f,g) \coloneqq \det(\nabla f, \nabla g)$, $f, g\colon \mathbb{D} \to \mathbb{R}$, satisfies the identities

$$\int_{\mathbb{S}^1} f\frac{\partial g}{\partial \tau} = \int_{\mathbb{D}} \mathrm{Jac}(f,g) \text{ and } \mathrm{Jac}(f,gh) = h\,\mathrm{Jac}(f,g) + g\,\mathrm{Jac}(f,h),$$

for all $f, g, h \in \mathrm{C}^1(\overline{\mathbb{D}};\mathbb{R})$, we find that

$$\begin{aligned}\int_{\mathbb{T}} \varphi'\zeta \equiv \int_{\mathbb{S}^1} \frac{\partial \varphi}{\partial \tau}\zeta = \int_{\mathbb{S}^1}\left[(u_1\zeta)\frac{\partial u_2}{\partial \tau} - (u_2\zeta)\frac{\partial u_1}{\partial \tau}\right] &= \int_{\mathbb{D}} [\mathrm{Jac}(\widetilde{u}_1\xi, \widetilde{u}_2) - \mathrm{Jac}(\widetilde{u}_2\xi, \widetilde{u}_1)] \\ &= 2\int_{\mathbb{D}} \xi\,\mathrm{Jac}\,\widetilde{u} + \int_{\mathbb{D}} [\widetilde{u}_1\,\mathrm{Jac}(\xi,\widetilde{u}_2) - \widetilde{u}_2\,\mathrm{Jac}(\xi,\widetilde{u}_1)] \\ &= 2\int_{\mathbb{D}} \xi\,\mathrm{Jac}\,\widetilde{u} + \int_{\mathbb{D}} \nabla\xi \wedge (\widetilde{u} \wedge \nabla\widetilde{u}) \coloneqq T_1(\zeta) + T_2(\zeta).\end{aligned}$$

We next prove that, for an appropriate choice of $\widetilde{u}$, the maps T_1 and T_2 satisfy items 2, 3 and 4 above.

Proof of item 2. We clearly have $T_2(1) = 0$ and $\int_{\mathbb{T}} \varphi' = 0$. This leads to $T_1(1) = 0$.

Proof of item 3. Let $M\zeta$ denote the maximal function of ζ. Recall the inequality

$$\sup_{0\leqslant r\leqslant 1} |\xi(r\omega)| \leqslant M\zeta(\omega), \quad \forall\, \omega \in \mathbb{S}^1 \tag{2.87}$$

(see Lemma 2.50). We have

$$\begin{aligned}\frac{1}{2}|T_1(\zeta)| &\leqslant \int_{\mathbb{D}} |\xi(x)||\,\mathrm{Jac}\,\widetilde{u}(x)|\,\mathrm{d}x = \int_{\mathbb{S}^1}\int_0^1 |\xi(r\omega)||\,\mathrm{Jac}\,\widetilde{u}(r\omega)|\,r\mathrm{d}r\mathrm{d}\omega \\ &\leqslant \int_{\mathbb{S}^1}\int_0^1 |\xi(r\omega)||\,\mathrm{Jac}\,\widetilde{u}(r\omega)|\,\mathrm{d}r\mathrm{d}\omega \leqslant \int_{\mathbb{S}^1} \sup_{0\leqslant r\leqslant 1}|\xi(r\omega)| \int_0^1 |\,\mathrm{Jac}\,\widetilde{u}(r\omega)|\,\mathrm{d}r\mathrm{d}\omega \\ &\leqslant \int_{\mathbb{S}^1} (M\zeta)(\omega)\int_0^1 |\,\mathrm{Jac}\,\widetilde{u}(r\omega)|\,\mathrm{d}r\mathrm{d}\omega =: \int_{\mathbb{S}^1} (M\zeta)(\omega)\varepsilon(\omega)\mathrm{d}\omega.\end{aligned}$$

Here, $\varepsilon(\omega) \coloneqq \int_0^1 |\operatorname{Jac} \widetilde{u}(r\omega)|\,\mathrm{d}r$. Applying Hölder's inequality, we obtain

$$\frac{1}{2}|T_1(\zeta)| \leqslant \|M\zeta\|_{\mathrm{L}^{(sp)'}}\|\varepsilon\|_{\mathrm{L}^{sp}} \lesssim \|\zeta\|_{\mathrm{L}^{(sp)'}}\|\varepsilon\|_{\mathrm{L}^{sp}} \tag{2.88}$$

by the maximal function theorem.

We now specify $\widetilde{u}$. Let v be the harmonic extension of u to $\mathbb{D}$. Let Π be as in (2.79). Then we set

$$\widetilde{u} \coloneqq \Pi \circ v. \tag{2.89}$$

The key estimate is

$$\|\varepsilon\|_{\mathrm{L}^{sp}} \lesssim |u|^{1/s}_{\mathrm{W}^{s,p}} \tag{2.90}$$

(see Lemma 2.51). This estimate, combined with (2.88), leads to $|T_1(\zeta)| \lesssim \|\zeta\|_{\mathrm{L}^{(sp)'}}|u|^{1/s}_{\mathrm{W}^{s,p}}$, i.e., item 3 holds.

Proof of item 4. We have

$$\begin{aligned} |T_2(\zeta)| &\leqslant \int_{\mathbb{D}} |\nabla\xi \wedge (\widetilde{u} \wedge \nabla\widetilde{u})|(x)\,\mathrm{d}x \leqslant \int_{\mathbb{D}} |\nabla\xi(x)||\nabla\widetilde{u}(x)|\,\mathrm{d}x \\ &= \int_{\mathbb{D}} \big(h(x)^{-1}|\nabla\xi(x)|\big)\,\big(h(x)|\nabla\widetilde{u}(x)|\big)\,\mathrm{d}x, \end{aligned}$$

where $h(x)$ will be specified afterwards. By Hölder's inequality we obtain

$$|T_2(\zeta)| \lesssim \Big(\int_{\mathbb{D}} h(x)^{-p'}|\nabla\xi(x)|^{p'}\,\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{D}} h(x)^{p}|\nabla\widetilde{u}(x)|^{p}\,\mathrm{d}x\Big)^{1/p}. \tag{2.91}$$

In order to estimate the right-hand side of (2.91), we rely on Lemma 2.55, which implies that

$$\int_{\mathbb{D}} (1-|x|)^{p-sp-1}|\nabla\widetilde{u}(x)|^p\,\mathrm{d}x \lesssim |u|^p_{\mathrm{W}^{s,p}} \tag{2.92}$$

and

$$\int_{\mathbb{D}} (1-|x|)^{sp'-1}|\nabla\xi(x)|^{p'}\,\mathrm{d}x \lesssim |\zeta|^{p'}_{\mathrm{W}^{1-s,p'}}. \tag{2.93}$$

By combining (2.92), (2.93) and (2.91) (applied with $h(x) \coloneqq (1-|x|)^{1-s-1/p}$), we obtain the desired estimate $|T_2(\zeta)| \lesssim |\zeta|_{\mathrm{W}^{1-s,p'}}|u|_{\mathrm{W}^{s,p}}$.

Step 2. *Proof of* (2.76) *in the general case.*

We assume now that $u \in \mathrm{W}^{s,p}(\mathbb{T};\mathbb{S}^1)$ and that $\varphi \in \mathrm{W}^{s,p}((0,1);\mathbb{R})$ is a phase of u. In order to use the result from **Step 1**, we proceed as follows. By extending φ by reflection and 2-periodicity we obtain a function ψ which belongs to $\mathrm{W}^{s,p}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ and is periodic. We define $w \coloneqq e^{\mathrm{i}\psi}$. We clearly have $|w|_{\mathrm{W}^{s,p}} \approx |u|_{\mathrm{W}^{s,p}}$. If ρ is a mollifier, then the maps $\psi_\varepsilon \coloneqq \psi * \rho_\varepsilon$ and $w_\varepsilon \coloneqq e^{\mathrm{i}\psi_\varepsilon}$ are smooth and verify $\psi_\varepsilon \to \psi$ and $w_\varepsilon \to w$ in $\mathrm{W}^{s,p}$, as $\varepsilon \to 0$. The convergence $w_\varepsilon \to w$ relies on the continuity of the map $\mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{R}) \ni \psi \mapsto e^{\mathrm{i}\psi} \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ when $0 < s < 1$ and $1 \leqslant p < \infty$ ([**51**, Theorem 1, Section 5.3.6]). By the previous step, we can write ψ_ε as the sum of two functions $\psi_{\varepsilon,1}$ and $\psi_{\varepsilon,2}$ in $\mathrm{W}^{s,p}(\mathbb{T};\mathbb{R})$ that satisfy the estimates

$$|\psi_{\varepsilon,1}|_{\mathrm{W}^{s,p}} \lesssim |w_\varepsilon|^{1/s}_{\mathrm{W}^{s,p}} \quad \text{and} \quad |\psi_{\varepsilon,2}|_{\mathrm{W}^{s,p}} \lesssim |w_\varepsilon|_{\mathrm{W}^{s,p}}.$$

Since $|w_\varepsilon|_{\mathrm{W}^{s,p}} \to |w|_{\mathrm{W}^{s,p}}$, we can apply Fatou's lemma to find some convergent subsequences $\psi_{j,1} \to \psi_1$ and $\psi_{j,2} \to \psi_2$ in L^p such that

$$|\psi_1|_{\mathrm{W}^{s,p}} \lesssim \liminf_j |\psi_{j,1}|_{\mathrm{W}^{s,p}} \quad \text{and} \quad |\psi_2|_{\mathrm{W}^{s,p}} \lesssim \liminf_j |\psi_{j,2}|_{\mathrm{W}^{s,p}}.$$

We thus have $\psi = \psi_1 + \psi_2$. Consequently, we may write $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1 \coloneqq \psi_1|_{(0,1)}$ belonging to $\mathrm{W}^{s,p}(\mathbb{T};\mathbb{R})$ and $\varphi_2 \coloneqq \psi_2|_{(0,1)}$ belonging to $\mathrm{W}^{s,p}(\mathbb{T};\mathbb{R})$ and satisfying the estimates

$$|\varphi_1|_{\mathrm{W}^{s,p}} \lesssim |w|^{1/s}_{\mathrm{W}^{s,p}} \approx |u|^{1/s}_{\mathrm{W}^{s,p}} \quad \text{and} \quad |\varphi_2|_{\mathrm{W}^{s,p}} \lesssim |w|_{\mathrm{W}^{s,p}} \approx |u|_{\mathrm{W}^{s,p}}. \qquad \square$$

We end this subsection by establishing the optimality of the estimates (2.75) and (2.76).

Proposition 2.16. — *The estimates* (2.75) *(when $n = 1$) and* (2.76) *(when $n \geqslant 2$) are optimal in the sense that* $\displaystyle\limsup_{|\varphi|_{\mathrm{W}^{s,p}} \to 0} \frac{|\varphi|_{\mathrm{W}^{s,p}}}{|u|_{\mathrm{W}^{s,p}}} > 0$ *and* $\displaystyle\limsup_{|u|_{\mathrm{W}^{s,p}} \to \infty} \frac{|\varphi|_{\mathrm{W}^{s,p}}}{|u|^{1/s}_{\mathrm{W}^{s,p}}} > 0$.

Proof. — When $n = 1$, the optimality of (2.75) "at ∞" was obtained by Merlet [**38**, Theorem 1.1]. We reproduce here its argument. Let $f \in \mathrm{C}^\infty_c((0,1);[0,1))$ be such that $f \not\equiv 0$. Define, for $j \geqslant 1$, $\varphi_j \coloneqq jf$ and $u_j \coloneqq e^{\mathrm{i}\varphi_j}$. Clearly, we have

$$|\varphi_j|_{\mathrm{W}^{s,p}} = j|f|_{\mathrm{W}^{s,p}} \approx j. \tag{2.94}$$

In computing $|u_j|_{\mathrm{W}^{s,p}}$, we use the estimates

$$|u_j(x) - u_j(y)| \approx j|f(x) - f(y)| \qquad \text{when} \qquad |x-y| < 1/j,$$
$$\int_0^{1-h} |f(x+h) - f(x)|^a \,\mathrm{d}x \approx |h|^a, \qquad \forall\, h \in \left(0, \frac{1}{2}\right) \qquad (\text{with } a \in \mathbb{R} \text{ fixed}),$$

and

$$|u_j(x) - u_j(y)| \lesssim 1 \qquad \text{when} \qquad |x-y| \geqslant 1/j.$$

Thus we have

$$\begin{aligned} j^{sp} &\approx j^p \iint_{|x-y|<1/j} \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{1+(s-1)p}} \lesssim |u_j|^p_{\mathrm{W}^{s,p}} \\ &\lesssim j^p \iint_{|x-y|<1/j} \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{1+(s-1)p}} + \iint_{|x-y|>1/j} \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{1+sp}} \approx j^{sp}. \end{aligned} \tag{2.95}$$

In particular, we have $|u_j|^p_{\mathrm{W}^{s,p}} \to \infty$ when $j \to \infty$. Moreover, (2.95) together with (2.94) yield $|u_j|^{1/s}_{\mathrm{W}^{s,p}} \approx |\varphi_j|_{\mathrm{W}^{s,p}}$.

The above example extends to higher dimension as in the **Step 3** of the proof of Theorem 2.10.

The optimality "at zero" is obvious since $|e^{\mathrm{i}\varphi}|_{\mathrm{W}^{s,p}} \leqslant |\varphi|_{\mathrm{W}^{s,p}}$ for any $\varphi \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{R})$. $\square$

2.7.5. Further thoughts when $sp < 1$. —

2.7.5.1. Existence of bounded phases and the sum-intersection property. — We address here the following question (also discussed in [**17**]).

Question (Q) Let $0 < s < 1$, $1 \leqslant p < \infty$ be such that $sp < 1$. Let $u \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$. Is there some $\varphi \in \mathrm{W}^{s,p} \cap \mathrm{L}^\infty(\mathbb{T}^n; \mathbb{R})$ such that $u = e^{\mathrm{i}\varphi}$?

The motivation behind this question is the following. The phase φ whose construction is described in the introduction depends only on u, not on s or p. This has the following consequence. Let $0 \leqslant \theta < 1$, $0 < s < 1$ and $1 \leqslant p < \infty$ be such that $sp < 1$. Let $u \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$. Then u belongs to all the spaces $\mathrm{W}^{\theta s,p/\theta}(\mathbb{T}^n; \mathbb{S}^1)$ (by the Gagliardo-Nirenberg embeddings) and thus $\varphi \in \mathrm{W}^{\theta s,p/\theta}$, $\forall\, \theta \in (0,1]$. We find that

$$\varphi \in \bigcap_{0<\theta\leqslant 1} \mathrm{W}^{\theta s,p/\theta} \subset \mathrm{W}^{s,p} \cap \left(\bigcap_{q<\infty} \mathrm{L}^q\right).$$

It is then natural to ask whether the above conclusion can be improved to $\varphi \in \mathrm{W}^{s,p} \cap \mathrm{L}^\infty$.

We start by noting that the answer to (Q) is positive when $p = 1$. Indeed, an inspection of the proof of Proposition 2.6 shows that the phase constructed there is bounded.

We next turn to the relevant range $0 < s < 1$, $1 < p < \infty$, $sp < 1$. Our main result here is the reduction of (Q) to a sum-intersection property of function spaces. In order to describe this property, we start with a very simple case which requires no technology. If $f \in \mathrm{L}^2$, then $f \in \mathrm{L}^1 + \mathrm{L}^\infty$ $\big($since $[\mathrm{L}^1, \mathrm{L}^\infty]_{1/2} = \mathrm{L}^2\big)$ and thus $\mathrm{L}^2 \subset \mathrm{L}^1 + \mathrm{L}^\infty$. Thus each map $f \in \mathrm{L}^2$ splits as $f = f_1 + f_2$, with $f_1 \in \mathrm{L}^1$ and $f_2 \in \mathrm{L}^\infty$. But more can be said. Indeed, we have $f = f_1 + f_2$, with $f_1 \coloneqq f\mathbb{1}_{\{|f|>1\}} \in \mathrm{L}^2 \cap \mathrm{L}^1$ and $f_2 \coloneqq f\mathbb{1}_{\{|f|\leqslant 1\}} \in \mathrm{L}^2 \cap \mathrm{L}^\infty$. Thus $\mathrm{L}^2 = (\mathrm{L}^2 \cap \mathrm{L}^1) + (\mathrm{L}^2 \cap \mathrm{L}^\infty)$. This is the sum-intersection property for the triple $(\mathrm{L}^2, \mathrm{L}^1, \mathrm{L}^\infty)$. This property also extends to other function spaces. Here is an example from [**40**]. If $\sigma > 1$ is not an integer and $p > \sigma$, then

$$\mathrm{W}^{1,\sigma} = \big(\mathrm{W}^{1,\sigma} \cap \mathrm{W}^{\sigma/p,p}\big) + \big(\mathrm{W}^{1,\sigma} \cap \mathrm{W}^{\sigma,1}\big).$$

We are now ready to reformulate (Q).

Proposition 2.17. — *(Q) holds if and only if (R) holds, where (R) is the property*

$$(R) \qquad \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{R}) = (\mathrm{W}^{s,p} \cap \mathrm{L}^\infty)(\mathbb{T}^n; \mathbb{R}) + \big(\mathrm{W}^{s,p} \cap \mathrm{W}^{sp,1}\big)(\mathbb{T}^n; \mathbb{R}).$$

Proof. — We may assume that $p > 1$, since both (Q) and (R) hold when $p = 1$.

Implication "$(Q) \Rightarrow (R)$".
Let $\varphi \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{R})$. Let $u \coloneqq e^{\mathrm{i}\varphi}$. Consider some $\psi \in \mathrm{W}^{s,p} \cap \mathrm{L}^\infty(\mathbb{T}^n; \mathbb{R})$ such that $u = e^{\mathrm{i}\psi}$. Then $\varphi = \psi + 2\pi f$, where $f \coloneqq \varphi - \psi 2\pi \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{Z})$. The following is a straightforward inequality. If $0 < s < 1$, $1 \leqslant p < \infty$ and if $f \in \mathrm{W}^{s,p}$ is integer-valued, then

$$(2.96) \qquad |f|_{\mathrm{W}^{sp,1}} \leqslant |f|^p_{\mathrm{W}^{s,p}}.$$

Using (2.96), we obtain that $\varphi = \psi + 2\pi f$, with $\psi \in \mathrm{W}^{s,p} \cap \mathrm{L}^\infty$ and $2\pi f \in \mathrm{W}^{s,p} \cap \mathrm{W}^{sp,1}$. Therefore, (R) holds.

Implication "$(R) \Rightarrow (Q)$".
Let $u \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$. Let $\varphi \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{R})$ be such that $u = e^{\mathrm{i}\varphi}$. Write $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1 \in \mathrm{W}^{s,p} \cap \mathrm{L}^\infty$ and $\varphi_2 \in \mathrm{W}^{s,p} \cap \mathrm{W}^{sp,1}$. Set $v \coloneqq e^{\mathrm{i}\varphi_2} \in \mathrm{W}^{sp,1}$. Then $v = e^{\mathrm{i}\varphi_3}$ for some $\varphi_3 \in \mathrm{W}^{sp,1} \cap \mathrm{L}^\infty$ (by the proof of Proposition 2.6). By the Gagliardo-Nirenberg embeddings, we have $\varphi_3 \in \mathrm{W}^{s,p} \cap \mathrm{L}^\infty$. Thus $u = e^{\mathrm{i}\psi}$, where $\psi \coloneqq \varphi_1 + \varphi_3 \in \mathrm{W}^{s,p} \cap \mathrm{L}^\infty$. □

We do not know whether (R) holds. It is easy to see that a weaker form of (R), where L^∞ is replaced by the slightly larger Besov space $B^0_{\infty,\infty}$, is valid:

$$\mathrm{W}^{s,p} = \left(\mathrm{W}^{s,p} \cap B^0_{\infty,\infty}\right) + \left(\mathrm{W}^{s,p} \cap \mathrm{W}^{sp,1}\right)$$

(see Lemma 2.56).

2.7.5.2. Lifting via the factorization method. — In this subsection, we propose a new lifting construction in the case where $sp < 1$. Our method relies on three ingredients:

- The factorization method (explained in Subsection 2.7.4, and used in the proof of Theorem 2.13).
- The averaging method of Dávila and Ignat [**23**] (which proved useful in Subsection 2.7.2, in the proof of Proposition 2.6).
- The theory of weighted Sobolev spaces, due among others to Uspenskiĭ [**54**] (for the results we use here, see also [**37**, Section 10.1.1, Theorem 1, p. 512] and the comprehensive discussion in [**44**]).

Let us explain the construction. Let $u\colon \mathbb{T}^n \to \mathbb{S}^1$. We first extend u to $\mathbb{R}^n$ as explained in the proof of Theorem 2.13, and define φ_1 as in (2.78). Recall that $\varphi_1 \in \mathrm{W}^{s,p}$ (Lemma 2.14). The key is the following substitute of Lemma 2.15.

Lemma 2.18. — *Let $1 \leqslant p < \infty$ and $0 < s < 1$ be such that $sp < 1$. Let $u \in \mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ and let φ_1 be as in* (2.78)*. Then we have $u\,e^{-\mathrm{i}\varphi_1} \in \mathrm{W}^{sp,1}(\mathbb{T}^n;\mathbb{S}^1)$.*

Assuming Lemma 2.18 proved for the moment, we complete the construction of a phase of u as follows. Set $v \coloneqq u\,e^{-\mathrm{i}\varphi_1}$. Since the map v belongs to $\mathrm{W}^{sp,1}$, we find that v has a phase φ_2 in the space $\mathrm{W}^{sp,1} \cap \mathrm{L}^\infty$ (by the proof of the Proposition 2.6). The Gagliardo-Nirenberg embeddings and the fact that φ_2 belongs to $\mathrm{W}^{sp,1} \cap \mathrm{L}^\infty$ imply that we also have $\varphi_2 \in \mathrm{W}^{s,p}$. In conclusion, $\varphi \coloneqq \varphi_1 + \varphi_2$ is a $\mathrm{W}^{s,p}$ phase of u.

It remains to proceed to the

Proof of Lemma 2.18. — A first ingredient of the proof is the following flat version of [**13**, Lemma 1.3]. For a related result, see Lemma 2.52. Let w be given by (2.77). For $x \in \mathbb{R}^n$, set

$$\lambda(x) \coloneqq \inf\left\{\varepsilon > 0 \,\middle|\, |w(x,\varepsilon)| = \frac{1}{2}\right\}. \tag{2.97}$$

Then λ satisfies

$$\int_{(-2,3)^n} \frac{1}{\lambda^{sp}(x)}\,\mathrm{d}x \lesssim |u|^p_{\mathrm{W}^{s,p}} + 1. \tag{2.98}$$

Estimate (2.98) is established in [**40**]. Alternatively, (2.98) can be obtained by adapting the proof of Lemma 2.52.

A second ingredient is provided by the following local estimate in the spirit of the theory of weighted Sobolev spaces.

Lemma 2.19. — *Let $0<\sigma<1$. Let $U\colon \mathbb{T}^n\times(0,\infty)\to\mathbb{C}$ be a smooth map. Assume that*

$$(2.99)\quad f(x):=\lim_{\varepsilon\to 0}U(x,\varepsilon)\quad \textit{exists for a.e. } x\in\mathbb{T}^n.$$

Then

$$(2.100)\quad |f|_{W^{\sigma,1}(\mathbb{T}^n)}\lesssim\int_{\mathbb{T}^n\times(0,1)}\varepsilon^{-\sigma}|\nabla U(x,\varepsilon)|\,dx d\varepsilon.$$

The proof of Lemma 2.19 is postponed to Subsection 2.7.7.6.

We will apply Lemma 2.19 with $\sigma:=sp$ and $U(x,\varepsilon):=\Pi\circ w(x,\varepsilon)\,e^{-i\psi(x,\varepsilon)}$. Here, w is as in (2.77), Π satisfies (2.79), and we set

$$(2.101)\quad \psi(x,\varepsilon):=-\int_{\varepsilon}^{\infty}\Pi\circ w(x,t)\wedge\frac{\partial}{\partial t}\Pi\circ w(x,t)\,dt,\qquad \forall\, x\in\mathbb{R}^n,\forall\,\varepsilon>0.$$

We now explain how these ingredients lead to the conclusion of Lemma 2.18.

Step 1. *U is smooth and* (2.99) *holds with $f:=u\,e^{-i\varphi_1}$.*

Indeed, since u equals a constant C in the set $\mathbb{R}^n\setminus(-4,5)^n$, we have

$$(2.102)\quad w(x,\varepsilon)=C+\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\rho\left(\frac{x-y}{\varepsilon}\right)[u(y)-C]\,dy=C+\frac{1}{\varepsilon^n}\int_{(-4,5)^n}\rho\left(\frac{x-y}{\varepsilon}\right)[u(y)-C]\,dy.$$

On the other hand, a straightforward induction on $|\alpha|$ leads to

$$(2.103)\quad \partial^\alpha\left(\frac{1}{\varepsilon^n}\rho\left(\frac{x-y}{\varepsilon}\right)\right)=O\left(\frac{1}{\varepsilon^{n+|\alpha|}}\right),\qquad \forall\,\alpha\in\mathbb{N}^{n+1}.$$

By combining (2.102) with (2.103) and with the fact that u is bounded, we find that

$$(2.104)\quad \partial^\alpha w(x,\varepsilon)=\int_{(-4,5)^n}\partial^\alpha\left(\frac{1}{\varepsilon^n}\rho\left(\frac{x-y}{\varepsilon}\right)\right)[u(y)-C]\,dy=O\left(\frac{1}{\varepsilon^{n+|\alpha|}}\right),\ \forall\,\alpha\in\mathbb{N}^{n+1}\setminus\{0\}.$$

In view of (2.104), we obtain by induction on $|\alpha|\geqslant 1$ that

$$(2.105)\quad \partial^\alpha(\Pi\circ w)(x,\varepsilon)=O\left(\frac{1}{\varepsilon^{n+|\alpha|}}\right)+O\left(\frac{1}{\varepsilon^{|\alpha|n+|\alpha|}}\right),\forall\,\alpha\in\mathbb{N}^{n+1}\setminus\{0\}.$$

This shows that ψ defined by (2.101) is smooth, and thus so is U. For further use, we also note that all derivatives of ψ are obtained by differentiating under the integral sign.

On the other hand, by Lebesgue's differentiation theorem we have $\lim\limits_{\varepsilon\to 0}w(x,\varepsilon)=u(x)$ for a.e. $x\in\mathbb{R}^n$. In addition, Lemma 2.14 *1* implies that $\lim\limits_{\varepsilon\to 0}\psi(x,\varepsilon)=\varphi_1(x)$ for a.e. $x\in\mathbb{T}^n$. We find that $\lim\limits_{\varepsilon\to 0}U(x,\varepsilon)=u(x)\,e^{-i\varphi_1(x)}$ for a.e. $x\in\mathbb{T}^n$.

Step 2. *Basic estimates.*

Let us note the fact that the inequality $|u|\leqslant 2$ implies that, in addition to (2.104), we have

$$(2.106)\quad |\partial^\alpha w(x,\varepsilon)|\lesssim\frac{1}{\varepsilon^{|\alpha|}},\qquad \forall\,\alpha\in\mathbb{N}^{n+1}.$$

In turn, (2.106) and formulas (2.77) and (2.79) lead, by induction on $|\alpha|$, to

$$|\partial^\alpha(\Pi\circ w)(x,\varepsilon)|\lesssim\frac{1}{\varepsilon^{|\alpha|}},\quad \forall\,\alpha\in\mathbb{N}^{n+1}. \tag{2.107}$$

Finally, (2.107) combined with the definition (2.101) of ψ leads to

$$|\partial^\alpha U(x,\varepsilon)|\lesssim\frac{1}{\varepsilon^{|\alpha|}},\quad \forall\,\alpha\in\mathbb{N}^{n+1}. \tag{2.108}$$

Step 3. *The role of* $\lambda(x)$.

Let $\lambda(x)$ be as in (2.97). In this step, we establish several identities valid at a point (x,ε) with $\varepsilon<\lambda(x)$.

To start with, it follows from the definition (2.97) of $\lambda(x)$ and from (2.79) that

$$|\Pi\circ w(x,\varepsilon)|\equiv 1\ \text{ in the open set }\ \mathcal{V}:=\{(x,\varepsilon)\in\mathbb{R}^n\times(0,\infty)\mid 0<\varepsilon<\lambda(x)\}\,. \tag{2.109}$$

By differentiating the identity (2.109), we find that

$$\nabla(\Pi\circ w)(x,\varepsilon)\perp\Pi\circ w(x,\varepsilon),\quad \forall\,(x,\varepsilon)\in\mathcal{V}. \tag{2.110}$$

By combining (2.110) with the identity

$$y=\mathrm{i}\omega(\omega\wedge y),\quad \forall\,y\in\mathbb{C},\ \forall\,\omega\in\mathbb{S}^1\text{ such that } y\perp\omega,$$

we find that

$$\nabla(\Pi\circ w)(x,\varepsilon)=\mathrm{i}(\Pi\circ w(x,\varepsilon))\,[\Pi\circ w(x,\varepsilon))\wedge(\nabla(\Pi\circ w)(x,\varepsilon))]\quad\text{in }\mathcal{V}. \tag{2.111}$$

On the other hand, (2.110) implies that in $\mathcal{V}$ the partial derivatives of $\Pi\circ w$ are mutually parallel. This leads to

$$\left(\frac{\partial}{\partial\varepsilon}(\Pi\circ w)\right)\wedge\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)(x,\varepsilon)=0\quad\text{in }\mathcal{V},\ \forall\,j\in[\![1,n]\!]. \tag{2.112}$$

We are now in position to compute ∇U in $\mathcal{V}$.

First, using (2.101) and (2.111) we find that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}U(x,\varepsilon)&=\frac{\partial}{\partial\varepsilon}(\Pi\circ w)(x,\varepsilon)e^{-\mathrm{i}\psi(x,\varepsilon)}-\mathrm{i}\frac{\partial}{\partial\varepsilon}\psi(x,\varepsilon)\Pi\circ w(x,\varepsilon)\,e^{-\mathrm{i}\psi(x,\varepsilon)}\\&=\left(\frac{\partial}{\partial\varepsilon}(\Pi\circ w)(x,\varepsilon)-\mathrm{i}\Pi\circ w(x,\varepsilon)\left[(\Pi\circ w)\wedge\left(\frac{\partial}{\partial\varepsilon}(\Pi\circ w)\right)\right](x,\varepsilon)\right)\\&\quad\times e^{-\mathrm{i}\psi(x,\varepsilon)}=0\quad\text{in }\mathcal{V}.\end{aligned} \tag{2.113}$$

We next note that an integration by parts combined with (2.101) and (2.112) leads to

$$
\begin{aligned}
\frac{\partial}{\partial x_j}\psi(x,\varepsilon) &= -\int_{\varepsilon}^{\infty}(\Pi\circ w)\wedge\left(\frac{\partial^2}{\partial t\partial x_j}(\Pi\circ w)\right)(x,t)\,\mathrm{d}t\\
&\quad-\int_{\varepsilon}^{\infty}\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)\wedge\left(\frac{\partial}{\partial t}(\Pi\circ w)\right)(x,t)\,\mathrm{d}t\\
&=(\Pi\circ w)\wedge\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)(x,\varepsilon)\\
&\quad-2\int_{\varepsilon}^{\infty}\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)\wedge\left(\frac{\partial}{\partial t}(\Pi\circ w)\right)(x,t)\,\mathrm{d}t\\
&=(\Pi\circ w)\wedge\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)(x,\varepsilon)\\
&\quad-2\int_{\lambda(x)}^{\infty}\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)\wedge\left(\frac{\partial}{\partial t}(\Pi\circ w)\right)(x,t)\,\mathrm{d}t\quad\text{in }\mathcal{V}.
\end{aligned}
\tag{2.114}
$$

A calculation similar to the one leading to (2.113) yields (using (2.114))

$$
\nabla_x U(x,\varepsilon)=2\mathrm{i}U(x,\varepsilon)\int_{\lambda(x)}^{\infty}(\nabla_x(\Pi\circ w))\wedge\left(\frac{\partial}{\partial t}(\Pi\circ w)\right)(x,t)\,\mathrm{d}t.
\tag{2.115}
$$

Step 4. *Estimate of* $\dfrac{\partial U}{\partial\varepsilon}$.

By combining (2.108) with (2.113), we find that

$$
\int_{\mathbb{T}^n\times(0,\infty)}\varepsilon^{-sp}\left|\frac{\partial}{\partial\varepsilon}U(x,\varepsilon)\right|\mathrm{d}x\mathrm{d}\varepsilon\lesssim\int_{\mathbb{T}^n}\int_{\lambda(x)}^{\infty}\varepsilon^{-sp-1}\,\mathrm{d}\varepsilon\mathrm{d}x\lesssim\int_{\mathbb{T}^n}\frac{1}{\lambda(x)^{sp}}\,\mathrm{d}x.
\tag{2.116}
$$

Step 5. *Estimate of* $\nabla_x U$.

This time, (2.108) combined with (2.115) and with the fact that $sp<1$ leads to

$$
\begin{aligned}
\int_{\mathbb{T}^n\times(0,\infty)}\varepsilon^{-sp}\,|\nabla_x U(x,\varepsilon)|\,\mathrm{d}x\mathrm{d}\varepsilon &\lesssim\int_{\mathbb{T}^n}\int_{\lambda(x)}^{\infty}\varepsilon^{-sp-1}\,\mathrm{d}\varepsilon\mathrm{d}x\\
&\quad+\int_{\mathbb{T}^n}\left(\int_0^{\lambda(x)}\varepsilon^{-sp}\,\mathrm{d}\varepsilon\right)\left(\int_{\lambda(x)}^{\infty}\frac{1}{t^2}\,\mathrm{d}t\right)\mathrm{d}x\\
&\lesssim\int_{\mathbb{T}^n}\frac{1}{\lambda(x)^{sp}}\,\mathrm{d}x.
\end{aligned}
\tag{2.117}
$$

Step 6. *Final conclusion.*

By combining **Step 1** and Lemma 2.19 with estimates (2.98), (2.116) and (2.117), we find that $u\,e^{-\mathrm{i}\varphi_1}\in\mathrm{W}^{sp,1}(\mathbb{T}^n)$, which is the conclusion of Lemma 2.18. □

2.7.6. Another application of the averaging method. Proof of Theorem 2.4. — In this subsection, we prove a quantitative version of Theorem 2.4. We start with the case $n = 1$, which is easier to follow. In this case, the main ingredient is Proposition 2.20. Once this proposition is obtained, the one dimensional case follows easily; see the proof of Theorem 2.21.

Our first result in this subsection is a sort of averaged "discrete" semi-norm estimate.

Proposition 2.20. — *Let* $0 < s < 1$, $1 \leqslant p < \infty$ *and* $f \in \mathrm{W}^{s,p}(\mathbb{T})$. *Then*

$$\sum_{j\geqslant 1} 2^{spj} \left\|(\tau_{2^{-j}} - \mathrm{id})\, f\right\|^p_{\mathrm{L}^p(\mathbb{T})} \lesssim \left[\frac{1}{s}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p \int_{\mathbb{T}} Y\left(f^y\right) \mathrm{d}y.$$

Proof. — By Lebesgue's differentiation theorem we have, for a.e. $x \in \mathbb{T}$,

$$\begin{aligned} f(x) = \lim_{k\to\infty} \fint_x^{x+2^{-k}} f(z)\,\mathrm{d}z &= \sum_{k\geqslant 1}\left(\fint_x^{x+2^{-k}} f(z)\,\mathrm{d}z - \fint_x^{x+2^{1-k}} f(z)\,\mathrm{d}z\right) + \int_{\mathbb{T}} f(z)\,\mathrm{d}z \\ &=: \sum_{k\geqslant 1} \delta_{2^{-k}} f(x) + \int_{\mathbb{T}} f(z)\,\mathrm{d}z. \end{aligned} \tag{2.118}$$

Here, $\delta_\varepsilon f(x) \coloneqq \fint_x^{x+\varepsilon} f(z)\,\mathrm{d}z - \fint_x^{x+2\varepsilon} f(z)\,\mathrm{d}z$.

Let $j \geqslant 1$. Applying the operator $\tau_{2^{-j}} - \mathrm{id}$ to the identity (2.118), we obtain

$$(\tau_{2^{-j}} - \mathrm{id})\, f(x) = (\tau_{2^{-j}} - \mathrm{id})\left(\sum_{k\geqslant 1} \delta_{2^{-k}} f(x)\right), \quad \text{for a.e. } x \in \mathbb{T}.$$

By Minkowski's inequality and the above estimate, we obtain that

$$\left\|(\tau_{2^{-j}} - \mathrm{id})\, f\right\|_{\mathrm{L}^p} \leqslant \sum_{k\geqslant 1} \left\|(\tau_{2^{-j}} - \mathrm{id})\ \delta_{2^{-k}} f\right\|_{\mathrm{L}^p}. \tag{2.119}$$

We split the sum in (2.119) as $\sum_{k\geqslant 1} \cdots = \sum_{1\leqslant k\leqslant j-\ell} \cdots + \sum_{k\geqslant j-\ell+1} \cdots =: S_1 + S_2$ (with ℓ integer to be determined later). On the one hand, we estimate S_1 via Lemma 2.34. On the other hand, we estimate S_2 using the trivial inequality

$$\|(\tau_h - \mathrm{id})g\|_{\mathrm{L}^p} \leqslant 2\|g\|_{\mathrm{L}^p}. \tag{2.120}$$

By combining (2.119), Lemma 2.34 and (2.120), we obtain

$$\left\|(\tau_{2^{-j}} - \mathrm{id})\, f\right\|_{\mathrm{L}^p} \lesssim \sum_{1\leqslant k\leqslant j-\ell} 2^{k-j} \left\|(\tau_{2^{-k}} - \mathrm{id})\, f\right\|_{\mathrm{L}^p} + \sum_{k\geqslant j-\ell+1} \left\|\delta_{2^{-k}} f\right\|_{\mathrm{L}^p}.$$

Hence for every $j \geqslant 1$ we have

$$2^{sj} \left\|(\tau_{2^{-j}} - \mathrm{id})\, f\right\|_{\mathrm{L}^p} \lesssim \sum_{1\leqslant k\leqslant j-\ell} 2^{k-(1-s)j} \left\|(\tau_{2^{-k}} - \mathrm{id})\, f\right\|_{\mathrm{L}^p} + \sum_{k\geqslant j-\ell+1} 2^{sj} \left\|\delta_{2^{-k}} f\right\|_{\mathrm{L}^p}. \tag{2.121}$$

Raising the inequalities in (2.121) to the power p and summing over j we find

$$\begin{aligned} \sum_{j\geqslant 1} 2^{spj} \left\|(\tau_{2^{-j}} - \mathrm{id})\, f\right\|^p_{\mathrm{L}^p} &\lesssim \sum_{j\geqslant 1}\left(\sum_{1\leqslant k\leqslant j-\ell} 2^{k-(1-s)j} \left\|(\tau_{2^{-k}} - \mathrm{id})\, f\right\|_{\mathrm{L}^p}\right)^p \\ &\quad + \sum_{j\geqslant 1}\left(\sum_{k\geqslant j-\ell+1} 2^{sj} \left\|\delta_{2^{-k}} f\right\|_{\mathrm{L}^p}\right)^p. \end{aligned} \tag{2.122}$$

In what follows we use the notation $X_j \coloneqq 2^{sj}\,\|(\tau_{2^{-j}} - \mathrm{id})\,f\|_{\mathrm{L}^p}$ and $Y_k \coloneqq 2^{sk}\,\|\delta_{2^{-k}} f\|_{\mathrm{L}^p}$. In terms of X_j and Y_k, (2.122) reads

$$\sum_{j\geqslant 1} X_j^p \lesssim \sum_{j\geqslant 1}\left(\sum_{1\leqslant k\leqslant j-\ell} 2^{-(1-s)(j-k)} X_k\right)^p + \sum_{j\geqslant 1}\left(\sum_{k\geqslant j-\ell+1} 2^{s(j-k)} Y_k\right)^p. \tag{2.123}$$

In order to estimate the sums in the right-hand side of (2.123), we apply Corollary 2.26. By combining (2.123) with Corollary 2.26, we obtain

$$\sum_{j\geqslant 1} X_j^p \leqslant C_1\left(\frac{2^{-(1-s)\ell}}{1-s}\right)^p \sum_{k\geqslant 1} X_k^p + C_2\left(\frac{2^{s\ell}}{s}\right)^p Y_k^p.$$

Hence

$$\left[1 - C_1\left(\frac{2^{-(1-s)\ell}}{1-s}\right)^p\right]\sum_{j\geqslant 1} X_j^p \lesssim \left(\frac{2^{s\ell}}{s}\right)^p \sum_{k\geqslant 1} Y_k^p. \tag{2.124}$$

We may choose a fixed real M and an integer $\ell = \ell(s,p)$ such that

$$\ell = -\frac{\log_2(1-s)}{(1-s)} + \frac{M + o(1)}{1-s} \quad \text{and} \quad 1 - C_1\left(\frac{2^{-(1-s)\ell}}{1-s}\right)^p = \frac{1}{2} + o(1) \quad \text{as } s \nearrow 1. \tag{2.125}$$

With this choice of ℓ, (2.124) and (2.125) lead to

$$\begin{aligned}\sum_{j\geqslant 1} 2^{spj}\,\|(\tau_{2^{-j}} - \mathrm{id})\,f\|_{\mathrm{L}^p}^p &\leqslant \left(\frac{1}{2} + o(1)\right)\left(\frac{1}{s(1-s)^s}\; 2^{M+o(1)}\right)^{p/(1-s)} \sum_{k\geqslant 1} 2^{spk}\,\|\delta_{2^{-k}} f\|_{\mathrm{L}^p}^p \\ &\leqslant \left(\frac{1}{2} + o(1)\right)\left(\frac{1}{s(1-s)}\; 2^{M+o(1)}\right)^{p/(1-s)} \sum_{k\geqslant 1} 2^{spk}\,\|\delta_{2^{-k}} f\|_{\mathrm{L}^p}^p \\ &\leqslant K_p\left[\frac{1}{s}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p \sum_{k\geqslant 1} 2^{spk}\,\|\delta_{2^{-k}} f\|_{\mathrm{L}^p}^p.\end{aligned} \tag{2.126}$$

By Lemma 2.31, we have

$$\|\delta_{2^{-k}} f\|_{\mathrm{L}^p}^p \leqslant 2\int_{\mathbb{T}} \|(f^y)_k - (f^y)_{k-1}\|_{\mathrm{L}^p}^p \,\mathrm{d}y. \tag{2.127}$$

We complete the proof of Proposition 2.20 by combining (2.126) with (2.127). □

We now state and prove a quantitative form of Theorem 2.4 with $n = 1$.

Theorem 2.21. — *Let $0 < s < 1$, $1 \leqslant p < \infty$ and $f \in \mathrm{W}^{s,p}(\mathbb{T})$. Then*

$$X(f) \lesssim \left[\frac{1}{s^2}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p \int_{\mathbb{T}} Y(f^y)\,\mathrm{d}y.$$

Proof. — We first note that

$$\begin{aligned} X(f) &= \int_{\mathbb{T}} \frac{\|(\tau_h - \mathrm{id})f\|_{\mathrm{L}^p}^p}{h^{1+sp}}\,\mathrm{d}h = \sum_{j\geqslant 1}\int_{2^{-j}}^{2^{1-j}} \frac{\|(\tau_h - \mathrm{id})f\|_{\mathrm{L}^p}^p}{h^{1+sp}}\,\mathrm{d}h \\ &\leqslant \sum_{j\geqslant 1} 2^{j(1+sp)}\int_{2^{-j}}^{2^{1-j}} \|(\tau_h - \mathrm{id})f\|_{\mathrm{L}^p}^p\,\mathrm{d}h.\end{aligned} \tag{2.128}$$

Let $j \geqslant 1$. For every $h \in \left[\dfrac{1}{2^j}, \dfrac{1}{2^{j-1}}\right)$ and $k \geqslant j$, we denote by $\varepsilon_k(h) \in \{0,1\}$ the k^{th} binary digit of h; thus

$$h = \sum_{k\geqslant j} \frac{\varepsilon_k(h)}{2^k} = \sum_{k\geqslant j,\ \varepsilon_k(h)=1} \frac{1}{2^k}. \tag{2.129}$$

We also note that

$$[0,1) \ni h \mapsto \|(\tau_h - \mathrm{id})f\|_{\mathrm{L}^p} \text{ is subadditive.} \tag{2.130}$$

(This follows from $|(\tau_{h+\ell} - \mathrm{id})f| \leqslant |\tau_h(\tau_\ell - \mathrm{id})f| + |(\tau_h - \mathrm{id})f|$ together with Minkowski's inequality.) From (2.129) and (2.130), we obtain that

$$\|(\tau_h - \mathrm{id})f\|_{\mathrm{L}^p} \leqslant \sum_{\substack{k\geqslant j\\ \varepsilon_k(h)=1}} \|(\tau_{2^{-k}} - \mathrm{id})\, f\|_{\mathrm{L}^p} \leqslant \sum_{k\geqslant j} \|(\tau_{2^{-k}} - \mathrm{id})\, f\|_{\mathrm{L}^p}, \quad \forall\, h \in \left[2^{-j}, 2^{1-j}\right). \tag{2.131}$$

Inserting (2.131) into (2.128), we find that

$$\begin{aligned} X(f) &\leqslant \sum_{j\geqslant 1} 2^{j(1+sp)} \int_{2^{-j}}^{2^{1-j}} \left(\sum_{k\geqslant j} \|(\tau_{2^{-k}} - \mathrm{id})\, f\|_{\mathrm{L}^p} \right)^p \mathrm{d}h \\ &= \sum_{j\geqslant 1} 2^{spj} \left(\sum_{k\geqslant j} \|(\tau_{2^{-k}} - \mathrm{id})\, f\|_{\mathrm{L}^p} \right)^p = \sum_{j\geqslant 1} \left[\sum_{k\geqslant j} 2^{s(j-k)} \left(2^{sk}\, \|(\tau_{2^{-k}} - \mathrm{id})\, f\|_{\mathrm{L}^p} \right) \right]^p. \end{aligned} \tag{2.132}$$

If we estimate the last sum in (2.132) via Corollary 2.26, we find that

$$X(f) \lesssim \frac{1}{s^p} \sum_{k\geqslant 1} 2^{spk}\, \|(\tau_{2^{-k}} - \mathrm{id})\, f\|_{\mathrm{L}^p}^p. \tag{2.133}$$

We complete the proof of Theorem 2.21 by combining (2.133) with Proposition 2.20. □

We now consider the case of an arbitrary n.

We start by adapting Proposition 2.20.

Proposition 2.22. — *Let $0 < s < 1$, $1 \leqslant p < \infty$ and $f \in \mathrm{W}^{s,p}(\mathbb{T}^n)$. Let $\{e_i\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$. Then, for every $i \in [\![1,n]\!]$,*

$$\sum_{j\geqslant 1} 2^{spj}\, \|(\tau_{2^{-j}e_i} - \mathrm{id})\, f\|_{\mathrm{L}^p(\mathbb{T}^n)}^p \lesssim \left[\frac{1}{s} \left(\frac{C_p}{1-s} \right)^{1/(1-s)} \right]^p \int_{\mathbb{T}^n} Y\left(f^y\right) \mathrm{d}y.$$

Proof. — We start by noting that, for a.e. $x \in \mathbb{T}$,

$$\begin{aligned} f(x) &= \lim_{k\to\infty} \fint_{x+(0,2^{-k})^n} f(z)\,\mathrm{d}z = \sum_{k\geqslant 1} \left(\fint_{x+(0,2^{-k})^n} f(z)\,\mathrm{d}z - \fint_{x+(0,2^{1-k})^n} f(z)\,\mathrm{d}z \right) \\ &\quad + \int_{\mathbb{T}} f(z)\,\mathrm{d}z =: \sum_{k\geqslant 1} \delta_{2^{-k}} f(x) + \int_{\mathbb{T}} f(z)\,\mathrm{d}z. \end{aligned} \tag{2.134}$$

Here, $\delta_\varepsilon f(x) := \displaystyle\fint_{x+(0,\varepsilon)^n} f(z)\,\mathrm{d}z - \fint_{x+(0,2\varepsilon)^n} f(z)\,\mathrm{d}z.$

Let $j \geqslant 1$ and $i \in [\![1,n]\!]$. Applying the operator $\tau_{2^{-j}e_i} - \mathrm{id}$ to the identity (2.134), and then Minkowski's inequality, we obtain

$$\|(\tau_{2^{-j}e_i} - \mathrm{id})\, f\|_{\mathrm{L}^p(\mathbb{T}^n)} \leqslant \sum_{k\geqslant 1} \|(\tau_{2^{-j}e_i} - \mathrm{id})\, \delta_{2^{-k}} f\|_{\mathrm{L}^p(\mathbb{T}^n)}. \tag{2.135}$$

As in the proof of Propostion 2.20, we split the sum in (2.135) as $\sum_{k\geqslant 1}\cdots = S_1 + S_2 \coloneqq \sum_{1\leqslant k\leqslant j-\ell}\cdots + \sum_{k\geqslant j-\ell+1}\cdots$, with ℓ an integer to be determined. We estimate S_1 via Lemma 2.35, and S_2 using the trivial inequality

$$\|(\tau_{he_i} - \mathrm{id})\, g\|_{\mathrm{L}^p(\mathbb{T}^n)} \leqslant 2\|g\|_{\mathrm{L}^p(\mathbb{T}^n)}. \tag{2.136}$$

Therefore, by combining (2.135), Lemma 2.35 and (2.136), we obtain

$$\|(\tau_{2^{-j}e_i} - \mathrm{id})\, f\|_{\mathrm{L}^p(\mathbb{T}^n)} \lesssim \sum_{1\leqslant k\leqslant j-\ell} 2^{k-j}\, \|(\tau_{2^{-k}e_i} - \mathrm{id})\, f\|_{\mathrm{L}^p(\mathbb{T}^n)} + \sum_{k\geqslant j-\ell+1} \|\delta_{2^{-k}} f\|_{\mathrm{L}^p(\mathbb{T}^n)}\,. \tag{2.137}$$

As in the proof of (2.122), this leads to

$$\begin{aligned}\sum_{j\geqslant 1} 2^{spj}\, \|(\tau_{2^{-j}e_i} - \mathrm{id})\, f\|^p_{\mathrm{L}^p(\mathbb{T}^n)} &\lesssim \sum_{j\geqslant 1}\left(\sum_{1\leqslant k\leqslant j-\ell} 2^{k-(1-s)j}\, \|(\tau_{2^{-k}e_i} - \mathrm{id})\, f\|_{\mathrm{L}^p(\mathbb{T}^n)}\right)^p \\ &\quad + \sum_{j\geqslant 1}\left(\sum_{k\geqslant j-\ell+1} 2^{sj}\, \|\delta_{2^{-k}} f\|_{\mathrm{L}^p(\mathbb{T}^n)}\right)^p.\end{aligned} \tag{2.138}$$

Using the notation $X^i_j \coloneqq 2^{sj}\, \|(\tau_{2^{-j}e_i} - \mathrm{id})\, f\|_{\mathrm{L}^p(\mathbb{T}^n)}$ and $Y_k \coloneqq 2^{sk}\, \|\delta_{2^{-k}} f\|_{\mathrm{L}^p(\mathbb{T}^n)}$, (2.138) reads

$$\sum_{j\geqslant 1}\left(X^i_j\right)^p \lesssim \sum_{j\geqslant 1}\left(\sum_{1\leqslant k\leqslant j-\ell} 2^{-(1-s)(j-k)} X^i_k\right)^p + \sum_{j\geqslant 1}\left(\sum_{k\geqslant j-\ell+1} 2^{s(j-k)} Y_k\right)^p. \tag{2.139}$$

As in the proof of (2.126), Corollary 2.26 combined with (2.139) leads, for an appropriate choice of ℓ, to

$$\sum_{j\geqslant 1} 2^{spj}\, \|(\tau_{2^{-j}e_i} - \mathrm{id})\, f\|^p_{\mathrm{L}^p(\mathbb{T}^n)} \leqslant K_p \left[\frac{1}{s}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p \sum_{k\geqslant 1} 2^{spk}\, \|\delta_{2^{-k}} f\|^p_{\mathrm{L}^p(\mathbb{T}^n)}\,. \tag{2.140}$$

We complete the proof of Proposition 2.22 by combining (2.140) with the inequality

$$\|\delta_{2^{-k}} f\|^p_{\mathrm{L}^p(\mathbb{T}^n)} \leqslant 2^n \int_{\mathbb{T}^n} \|(f^y)_k - (f^y)_{k-1}\|^p_{\mathrm{L}^p(\mathbb{T}^n)}\, \mathrm{d}y \tag{2.141}$$

(see Lemma 2.33). □

Proof of Theorem 2.4. — Let $f \in \mathrm{W}^{s,p}(\mathbb{T}^n)$. Since $[0,1)^n \ni v \mapsto \|(\tau_v - \mathrm{id})f\|_{\mathrm{L}^p(\mathbb{T}^n)}$ is subadditive, we can estimate $X(f)$ by

$$\begin{aligned} X(f) &= \int_{\mathbb{T}^n} \frac{\|(\tau_v - \mathrm{id})f\|^p_{\mathrm{L}^p(\mathbb{T}^n)}}{|v|^{n+sp}}\,\mathrm{d}v \lesssim \sum_{i=1}^n \int_{\mathbb{T}^n} \frac{\|(\tau_{v_ie_i} - \mathrm{id})f\|^p_{\mathrm{L}^p(\mathbb{T}^n)}}{|(v_1,\ldots,v_n)|^{n+sp}}\,\mathrm{d}v_1\ldots\mathrm{d}v_n \\ &\lesssim \sum_{i=1}^n \int_{\mathbb{T}} \frac{\|(\tau_{he_i} - \mathrm{id})f\|^p_{\mathrm{L}^p(\mathbb{T}^n)}}{h^{1+sp}}\,\mathrm{d}h \leqslant \sum_{i=1}^n \sum_{j\geqslant 1} 2^{j(1+sp)} \int_{2^{-j}}^{2^{1-j}} \|(\tau_{he_i} - \mathrm{id})f\|^p_{\mathrm{L}^p(\mathbb{T}^n)}\,\mathrm{d}h. \end{aligned} \tag{2.142}$$

In (2.142), we rely on Corollary 2.43 in order to justify the second inequality.

Following the proof of (2.133), we obtain, for every $i \in [\![1,n]\!]$, the estimate

$$\sum_{j\geqslant 1} 2^{j(1+sp)} \int_{2^{-j}}^{2^{1-j}} \|(\tau_{he_i} - \mathrm{id})f\|^p_{\mathrm{L}^p(\mathbb{T}^n)}\,\mathrm{d}h \lesssim \frac{1}{s^p}\sum_{k\geqslant 1} 2^{spk}\, \|(\tau_{2^{-k}e_i} - \mathrm{id})\, f\|^p_{\mathrm{L}^p(\mathbb{T}^n)}\,. \tag{2.143}$$

By combining (2.142) and (2.143), we find that

$$X(f) \lesssim \frac{1}{s^p}\sum_{i=1}^n\sum_{k\geqslant 1} 2^{spk}\, \|(\tau_{2^{-k}e_i} - \mathrm{id})\, f\|^p_{\mathrm{L}^p(\mathbb{T}^n)}\,. \tag{2.144}$$

Applying Proposition 2.22 to (2.144), we obtain

$$X(f) \lesssim \left[\frac{1}{s^2}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p \int_{\mathbb{T}^n} Y(f^y)\,\mathrm{d}y, \tag{2.145}$$

hence the conclusion. □

Remark 2.23. — *It would be interesting to obtain the analog of Theorem 2.4 when* $s \geqslant 1$*. Here is a hint suggesting that such an analog should exist. Using Fourier series, it is easy to see that the right-hand side of* (2.32) *converges when* $f \in \mathrm{W}^{1,2} = H^1$*, and that we have the estimate*

$$|f|^2_{\mathrm{W}^{1,2}} \lesssim \int_{\mathbb{T}^n} \sum_{j\geqslant 1} 2^{2j} \left\|(f^y)_j - (f^y)_{j-1}\right\|^2_{\mathrm{L}^2}\,\mathrm{d}y. \tag{2.146}$$

Here, we consider e.g. the semi-norm

$$\left|\sum_{m\in\mathbb{Z}^n} c_m\, e^{2\mathrm{i}\pi m\cdot x}\right|^2_{\mathrm{W}^{1,2}} = \sum_{m\in\mathbb{Z}^n} |m|^2 |c_m|^2.$$

The analog of (2.146) *for other values of* $s \geqslant 1$ *and* p *has not been investigated.*

Remark 2.24. — *The quantitative form of Theorem 2.4 is not optimal, at least when* $p = 1$*. Indeed, when* $p = 1$ *estimate* (2.145) *deteriorates exponentially fast when* $s \nearrow 1$*, while we know from estimate* (2.150) *that the growth is of the order of* $1/(1-s)$*. We do not know the optimal blow up rate when* $1 \leqslant p < \infty$ *and* $s \nearrow 1$*.*

2.7.7. Toolbox. — We present here the proofs of several auxiliary estimates used in the previous subsections.

2.7.7.1. Schur's criterion and applications. — The material presented in this subsection was mainly used in the proof of Theorem 2.3.

We start by recalling (a slight generalization of) Schur's condition – or Schur's criterion – on the boundedness of integral operators and by presenting some of its consequences of interest to us. For a further discussion on Schur's criterion, see e.g. [**31**, Appendix I].

Lemma 2.25. — *Let* (X,μ) *and* (Y,ν) *be two measure spaces, and let* p*,* q *be conjugated exponents. Consider the integral operator* T *associated with a measurable kernel* $\kappa: X\times Y \to \mathbb{C}$*, defined formally by*

$$Tu(x) = \int_Y \kappa(x,y)u(y)\,\mathrm{d}\nu(y), \quad \forall\, u: Y \to \mathbb{C}.$$

Let $f: X\times Y \to \mathbb{C}$ *be a measurable function on* X*, and set* $g(x) \coloneqq \left\|f(x,\cdot)\,|\kappa(x,\cdot)|^{1/q}\right\|_{\mathrm{L}^q(Y)}$*.*

If $M \coloneqq \operatorname*{ess\,sup}_y \left\|\dfrac{g}{f(\cdot,y)}\,|\kappa(\cdot,y)|^{1/p}\right\|_{\mathrm{L}^p(X)}$ *is finite, then* T *defines a bounded operator from* $\mathrm{L}^p(Y)$ *into* $\mathrm{L}^p(X)$*, with* $\|T\| \leqslant M$*.*

In particular (with the choice $f \equiv 1$*) we have*

$$\|T\| \leqslant M_1^{1/q} M_2^{1/p}, \tag{2.147}$$

where $M_1 \coloneqq \operatorname*{ess\,sup}_x \int_Y |\kappa(x,y)|\,\mathrm{d}\nu(y)$ *and* $M_2 \coloneqq \operatorname*{ess\,sup}_y \int_X |\kappa(x,y)|\,\mathrm{d}\mu(x)$*.*

Proof. — By a standard argument, it suffices to establish the bound $\|Tu\|_{\mathrm{L}^p} \leqslant M\|u\|_{\mathrm{L}^p}$ when κ, f and u are non negative. We assume that $p < \infty$; the case where $p = \infty$ is similar. By a suitable application of Hölder's inequality, we find that

$$\left(\int_Y \kappa(x,y)u(y)\,\mathrm{d}\nu(y)\right)^p = \left(\int_Y \left(f(x,y)\kappa^{1/q}(x,y)\right)\left(\kappa^{1/p}(x,y)\frac{u(y)}{f(x,y)}\right)\mathrm{d}\nu(y)\right)^p$$
$$\leqslant g^p(x)\int_Y \kappa(x,y)\frac{u^p(y)}{f^p(x,y)}\,\mathrm{d}\nu(y).$$

Therefore,

$$\int_X (Tu(x))^p\,\mathrm{d}\mu(x) \leqslant \int_Y\left(\int_X \frac{g^p(x)}{f^p(x,y)}\kappa(x,y)\,\mathrm{d}\mu(x)\right)u^p(y)\,\mathrm{d}\nu(y) \leqslant M^p\|u\|^p_{\mathrm{L}^p(Y)}.$$

The special case is obtained by noting that, when $f \equiv 1$, we have $g \leqslant M_1^{1/q}$, which implies that $M \leqslant M_1^{1/q}M_2^{1/p}$. □

By taking $f \equiv 1$ in the above lemma, we obtain the following consequence.

Corollary 2.26. — *Let $(\alpha_{j,k})_{j,k\geqslant 0}$ be an infinite matrix, and let $1 \leqslant p < \infty$. Consider the operator T formally defined by $T(x_k)_{k\geqslant 0} = \left(\sum_{k\geqslant 0}\alpha_{j,k}x_k\right)_{j\geqslant 0}$.*

If the quantity $M \coloneqq \sup_{i\geqslant 0}\left(\sum_{j=0}^{\infty}|\alpha_{j,i}|\left(\sum_{k=0}^{\infty}|\alpha_{j,k}|\right)^{p-1}\right)^{1/p}$ is finite, then T is continuous from ℓ^p into ℓ^p, with $\|T\| \leqslant M$.

In particular, we have, for $1 \leqslant p \leqslant \infty$,

$$\|T\| \leqslant \left(\sup_j \sum_k |\alpha_{j,k}|\right)^{1/q}\left(\sup_k \sum_j |\alpha_{j,k}|\right)^{1/p}.$$

We continue with a quantitative form of the equivalence $X(f) \sim Y(f) \sim Z(f)$ when $sp < 1$. Here, $X(F)$, $Y(f)$ and $Z(f)$ are given by (2.34) – (2.36). The next result and its proof follow closely [**10**, Appendix A].

Lemma 2.27. — *Let $0 < s < 1$ and $1 \leqslant p < \infty$. Let $f \in \mathrm{L}^p(\mathbb{T}^n)$, and let $X(f)$, $Y(f)$ and $Z(f)$ be as in* (2.34) – (2.36)*. Then*

(2.148) $s^pZ(f) \lesssim Y(f) \leqslant 2Z(f)$,

(2.149) $Z(f) \lesssim X(f)$

and, if $sp < 1$,

(2.150) $X(f) \lesssim \dfrac{1}{s^p(1-sp)^p}Y(f)$.

Proof. —
Step 1. *Proof of* (2.150).

We have that

(2.151) $$X(f) = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\frac{|(\tau_h - \mathrm{id})f(x)|^p}{|h|^{n+sp}}\,\mathrm{d}x\mathrm{d}h \leqslant \sum_{j=1}^{\infty}2^{(n+sp)j}\int_{|h|\in I_j}\|(\tau_h - \mathrm{id})f\|^p_{\mathrm{L}^p}\,\mathrm{d}h,$$

where $I_j = [2^{-j}, 2^{-(j-1)})$. Since $f_k \to f$ in $\mathrm{L}^p(\mathbb{T}^n)$, and $f_0 = \displaystyle\int_{\mathbb{T}^n} f$ is constant, we have

$$\|(\tau_h - \mathrm{id})f\|_{\mathrm{L}^p} = \left\|\sum_{k=1}^{\infty}(\tau_h - \mathrm{id})(f_k - f_{k-1})\right\|_{\mathrm{L}^p} \leqslant \sum_{k=1}^{\infty}\|(\tau_h - \mathrm{id})(f_k - f_{k-1})\|_{\mathrm{L}^p}. \tag{2.152}$$

We next invoke [**10**, Lemma A.2] in the following form:

Lemma 2.28. — *Let $f \in \mathscr{E}_k$, $j \geqslant 1$ and $h \in \mathbb{T}^n$ be such that $|h| < 2^{1-j}$. Then*

$$\|(\tau_h - \mathrm{id})f\|_{\mathrm{L}^p} \lesssim \beta_{j,k}\|f\|_{\mathrm{L}^p}, \tag{2.153}$$

where

$$\beta_{j,k} \coloneqq \begin{cases} 1, & \text{if } j \leqslant k \\ (2^{k-j})^{1/p}, & \text{if } j > k \end{cases}. \tag{2.154}$$

Step 1 completed.
Let $x_k \coloneqq 2^{sk}\,\|f_k - f_{k-1}\|_{\mathrm{L}^p}$, $\forall\, k \geqslant 1$, (so that $Y(f) = \|(x_k)_{k\geqslant 1}\|_{\ell^p}^p$) and set $\alpha_{j,k} \coloneqq 2^{s(j-k)}\beta_{j,k}$. We note that

$$\sum_j \alpha_{j,k} = \sum_k \alpha_{j,k} = \frac{1}{1-2^{-s}} + \frac{1}{2^{s-1/p}-1} \lesssim \frac{1}{s} + \frac{1}{1-sp} \lesssim \frac{1}{s(1-sp)}. \tag{2.155}$$

Next, Lemma 2.28 combined with (2.151) and (2.152) gives

$$X(f) \lesssim \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\alpha_{j,k}x_k\right)^p. \tag{2.156}$$

We obtain (2.150) by combining (2.156) with Corollary 2.26.

Step 2. *Proof of* (2.148).

Since $\|f_j - f_{j-1}\|_{\mathrm{L}^p} = \|E_j\,(f - f_{j-1})\|_{\mathrm{L}^p} \leqslant \|f - f_{j-1}\|_{\mathrm{L}^p}$, we find that

$$Y(f) \leqslant 2^{sp}Z(f) \leqslant 2Z(f).$$

On the other hand, we have

$$\|f - f_j\|_{\mathrm{L}^p} = \left\|\sum_{k\geqslant j+1}(f_k - f_{k-1})\right\|_{\mathrm{L}^p} \leqslant \sum_{k\geqslant j+1}\|f_k - f_{k-1}\|_{\mathrm{L}^p},$$

and thus

$$Z(f) \leqslant \sum_{j\geqslant 0}\left(\sum_{k\geqslant j+1}2^{-s(k-j)}x_k\right)^p \lesssim \frac{1}{s^p}Y(f)$$

(the last inequality following from Corollary 2.26).

Step 3. *Proof of* (2.149). By Hölder's inequality we have

$$\|f - f_j\|_{\mathrm{L}^p}^p \leqslant \int_{\mathbb{T}^n}\left(\fint_{Q_j(x)}|f(x) - f(y)|\,\mathrm{d}y\right)^p \mathrm{d}x \leqslant \int_{\mathbb{T}^n}\fint_{Q_j(x)}|f(x) - f(y)|^p\,\mathrm{d}y\mathrm{d}x.$$

Therefore,

$$Z(f) \leqslant \sum_{j\geqslant 0} 2^{(n+sp)j} \int_{\mathbb{T}^n} \int_{Q_j(x)} |f(x)-f(y)|^p \, \mathrm{d}y\mathrm{d}x$$
$$= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \left(|x-y|^{n+sp} \sum_{j\geqslant 0} 2^{(n+sp)j} \mathbb{1}_{Q_j(x)}(y) \right) \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y\mathrm{d}x.$$

In order to evaluate the above expression between brackets, we fix $x \neq y$ in $\mathbb{T}^n$ and we let k be such that $|x-y| \in I_k$. Then

$$|x-y|^{n+sp} \sum_{j\geqslant 0} 2^{(n+sp)j} \mathbb{1}_{Q_j(x)}(y) \leqslant |x-y|^{n+sp} \sum_{j=0}^{k-1} 2^{(n+sp)j} \lesssim 1,$$

which implies (2.149). □

For further use, let us recall the following cousin of Lemma 2.27 [**10**, Corollary A.1].

Lemma 2.29. — *Let $0 < s < 1$ and $1 \leqslant p < \infty$ be such that $sp < 1$. Let $f^j \colon \mathbb{T}^n \to \mathbb{R}$ be a sequence of functions such that $f^j \in \mathcal{E}_j$, $\forall\, j$ (we recall that $\mathcal{E}_j$ denotes the class of functions which are constant on each dyadic cube of $\mathcal{P}_j$). If*

$$\sum_{j\geqslant 1} 2^{spj} \left\| f^j - f^{j-1} \right\|_{\mathrm{L}^p}^p < \infty,$$

then f^j converges in L^p to a function $f \in \mathrm{W}^{s,p}$. In addition, we have

$$|f|_{\mathrm{W}^{s,p}}^p \lesssim \sum_{j\geqslant 1} 2^{spj} \left\| f^j - f^{j-1} \right\|_{\mathrm{L}^p}^p .$$

2.7.7.2. Estimates for averages. — The material in this subsection was used in the proofs of Theorems 2.3 and 2.4.

We start with a version of [**10**, (E.17)].

Lemma 2.30. — *Let f belong to $\mathscr{E}_k$, $\rho := \mathbb{1}_{(-1/2,1/2)^n}$, and $\rho_\varepsilon(x) := \varepsilon^{-n}\rho\left(\dfrac{x}{\varepsilon}\right)$, $\forall\, \varepsilon > 0$, $\forall\, x$. Let h satisfy $|h| < 2^{-j}$, where $j \geqslant k$. Then*

$$|\tau_h f - f| \leqslant 2^{2n+1} |f| * \rho_{2^{2-k}} \mathbb{1}_{A_{k,j}},$$

where

$$A_{k,j} := \left\{ x \in \mathbb{T}^n \,\middle|\, \mathrm{dist}(x, \partial Q) \leqslant 2^{-j} \text{ for some } Q \in \mathscr{P}_k \right\}.$$

Proof. — Since f is constant in each cube $Q \in \mathscr{P}_k$ and $|h| < 2^{-k}$, we have

$$(2.157)\quad |\tau_h f|(x) = \fint_{Q_k(x-h)} |f| \leqslant 2^{nk} \int_{B(x-h,2^{-k})} |f| \leqslant 2^{nk} \int_{B(x,2^{1-k})} |f|.$$

We note also that

$$(2.158)\quad |f| * \rho_{2^{2-k}}(x) = 2^{n(k-2)} \int_{B(x,2^{1-k})} |f|.$$

By combining (2.157) with (2.158) we obtain

$$(2.159)\quad |\tau_h f| \leqslant 2^{2n} |f| * \rho_{2^{2-k}}.$$

By letting $h \to 0$ in (2.159), we find that

$$(2.160)\quad |f| \leqslant 2^{2n}|f| * \rho_{2^{2-k}}.$$

By (2.159) and (2.160), we obtain

$$|\tau_h f - f| \leqslant |\tau_h f| + |f| \leqslant 2^{2n+1}|f| * \rho_{2^{2-k}}.$$

Now the conclusion follows by noting that, when x does not belong to $A_{k,j}$ we have $Q_k(x-h) = Q_k(x)$, and thus $\tau_h f(x) = f(x)$. $\square$

We next turn to Lemmas 2.31–2.35 which were used in Subsection 2.7.6.

Lemma 2.31. — *Let $1 \leqslant p < \infty$ and $f \in \mathrm{L}^p(\mathbb{T})$. Let δ_ε be the operator given by*

$$(2.161)\quad \delta_\varepsilon f(x) := \fint_0^\varepsilon f(x+z)\,\mathrm{d}z - \fint_0^{2\varepsilon} f(x+z)\,\mathrm{d}z.$$

Then, for every $k \geqslant 1$,

$$\int_{\mathbb{T}} \|(f^y)_k - (f^y)_{k-1}\|^p_{\mathrm{L}^p(\mathbb{T})}\,\mathrm{d}y \geqslant \frac{1}{2}\,\|\delta_{2^{-k}} f\|^p_{\mathrm{L}^p(\mathbb{T})}.$$

Recall that $f^y(x) = f(x-y)$.

Proof. — We first note that for every $x \in \mathbb{T}$, the dyadic cube (interval) of x of order k is given by

$$Q_k(x) = 2^{-k}\left[2^k x\right] + \left[0, 2^{-k}\right).$$

Note also that if x belongs to an interval of the form $J_{k,\ell} := [2^{1-k}\ell, 2^{-k}(2\ell+1))$ with $\ell \in [\![0, 2^{k-1}-1]\!]$, then we have $2^{1-k}[2^{k-1}x] = 2^{-k}\left[2^k x\right]$. Thus, for every such x and every $y \in \mathbb{T}$, we have

$$(2.162)\quad \begin{aligned}\fint_{Q_k(x)} f^y(z)\,\mathrm{d}z - \fint_{Q_{k-1}(x)} f^y(z)\,\mathrm{d}z &= \fint_0^{2^{-k}} f^y\left(z + 2^{-k}[2^k x]\right)\mathrm{d}z\\ &\quad - \fint_0^{2^{1-k}} f^y\left(z + 2^{1-k}[2^{k-1}x]\right)\mathrm{d}z = \delta_{2^{-k}} f^y\left(2^{-k}\left[2^k x\right]\right).\end{aligned}$$

We next note that $\delta_\varepsilon f^y(x) = \delta_\varepsilon f(x-y)$, $\forall\, x, y \in \mathbb{T}^n$. If $x \in J_{k,\ell}$, then by integrating (2.162) with respect to y we find that

$$(2.163)\quad \begin{aligned}\int_{\mathbb{T}} \left|\fint_{Q_k(x)} f^y(z)\,\mathrm{d}z - \fint_{Q_{k-1}(x)} f^y(z)\,\mathrm{d}z\right|^p \mathrm{d}y &= \int_{\mathbb{T}} \left|\delta_{2^{-k}} f^y\left(2^{-k}\left[2^k x\right]\right)\right|^p \mathrm{d}y\\ &= \int_{\mathbb{T}} \left|\delta_{2^{-k}} f\left(2^{-k}\left[2^k x\right] - y\right)\right|^p \mathrm{d}y\\ &= \int_{\mathbb{T}} |\delta_{2^{-k}} f(y)|^p\,\mathrm{d}y.\end{aligned}$$

We obtain the conclusion by integrating the left-hand side of (2.163) with respect to $x \in J_{k,\ell}$, $\forall\, \ell$. $\square$

Remark 2.32. — *It is not difficult to see that the following extension of* (2.162) *holds for every* $x \in \mathbb{T}$*:*

$$\left| \fint_{Q_k(x)} f^y(z)\,\mathrm{d}z - \fint_{Q_{k-1}(x)} f^y(z)\,\mathrm{d}z \right| = \left| \delta_{2^{-k}} f^y \left(2^{-k}\left[2^k x\right]\right)\right|.$$

Hence the conclusion of Lemma 2.31 can be improved to

$$\int_{\mathbb{T}} \|(f^y)_k - (f^y)_{k-1}\|^p_{\mathrm{L}^p(\mathbb{T})}\,\mathrm{d}y = \|\delta_{2^{-k}} f\|^p_{\mathrm{L}^p(\mathbb{T})}.$$

However, the advantage of the Lemma 2.31 stated as above is that its proof can be easily generalized to higher dimension.

Lemma 2.31 has the following n-dimensional analog.

Lemma 2.33. — *Let* $1 \leqslant p < \infty$ *and* $f \in \mathrm{L}^p(\mathbb{T}^n)$*. Let* δ_ε *be the operator given by*

$$\delta_\varepsilon f(x) := \fint_{(0,\varepsilon)^n} f(x+z)\,\mathrm{d}z - \fint_{(0,2\varepsilon)^n} f(x+z)\,\mathrm{d}z. \tag{2.164}$$

Then, for every $k \geqslant 1$*,*

$$\int_{\mathbb{T}^n} \|(f^y)_k - (f^y)_{k-1}\|^p_{\mathrm{L}^p(\mathbb{T}^n)}\,\mathrm{d}y \geqslant \frac{1}{2^n}\|\delta_{2^{-k}} f\|^p_{\mathrm{L}^p(\mathbb{T}^n)}.$$

The proof of Lemma 2.33 is identical to the proof of Lemma 2.31.

Lemma 2.34. — *Let* $f \in \mathrm{L}^p(\mathbb{T})$ *and let* δ_ε *the operator given by* (2.161)*. Then*

$$\|(\tau_h - \mathrm{id})\,\delta_\varepsilon f\|_{\mathrm{L}^p} \leqslant \frac{h}{\varepsilon}\|(\tau_\varepsilon - \mathrm{id})\,f\|_{\mathrm{L}^p}, \qquad \forall\, h \in [0,\varepsilon].$$

Proof. — Let $0 \leqslant h \leqslant \varepsilon$. For every $x \in [0,1)$, by the definition of δ_ε we have

$$\begin{aligned}
(\tau_h - \mathrm{id})\,\delta_\varepsilon f(x) &= \frac{1}{2\varepsilon}\left(2\int_{-h}^{\varepsilon-h} f(x+z)\,\mathrm{d}z - \int_{-h}^{2\varepsilon-h} f(x+z)\,\mathrm{d}z\right)\\
&\quad - \frac{1}{2\varepsilon}\left(2\int_0^{\varepsilon} f(x+z)\,\mathrm{d}z - \int_0^{2\varepsilon} f(x+z)\,\mathrm{d}z\right)\\
&= \frac{1}{2\varepsilon}\left(\int_{-h}^0 f(x+z)\,\mathrm{d}z - 2\int_{\varepsilon-h}^{\varepsilon} f(x+z)\,\mathrm{d}z + \int_{2\varepsilon-h}^{2\varepsilon} f(x+z)\,\mathrm{d}z\right)\\
&= \frac{1}{2\varepsilon}\left(\int_{-h}^0 f(x+z)\,\mathrm{d}z - \int_{\varepsilon-h}^{\varepsilon} f(x+z)\,\mathrm{d}z\right)\\
&\quad - \frac{1}{2\varepsilon}\left(\int_{\varepsilon-h}^{\varepsilon} f(x+z)\,\mathrm{d}z - \int_{2\varepsilon-h}^{2\varepsilon} f(x+z)\,\mathrm{d}z\right)\\
&= \frac{1}{2\varepsilon}\left[\int_{\varepsilon-h}^{\varepsilon} (\tau_\varepsilon - \mathrm{id}) f(x+z)\,\mathrm{d}z - \int_{2\varepsilon-h}^{2\varepsilon} (\tau_\varepsilon - \mathrm{id}) f(x+z)\,\mathrm{d}z\right].
\end{aligned}$$

Hence, for every $x \in \mathbb{T}$, we have

$$|(\tau_h - \mathrm{id})\,\delta_\varepsilon f(x)| \leqslant \frac{1}{2\varepsilon}\left[\int_{\varepsilon-h}^{\varepsilon} |(\tau_\varepsilon - \mathrm{id}) f(x+z)|\,\mathrm{d}z + \int_{2\varepsilon-h}^{2\varepsilon} |(\tau_\varepsilon - \mathrm{id}) f(x+z)|\,\mathrm{d}z\right]. \tag{2.165}$$

We note that for any $F \in \mathrm{L}^p(\mathbb{T})$ and for $\rho = \mathbb{1}_{(-1/2,1/2)}$ we have $F * \rho_h(t) = \dfrac{1}{h}\displaystyle\int_{t-h/2}^{t+h/2} F(z)\,\mathrm{d}z$. Thus

$$\begin{aligned}(2.166)\quad &\frac{1}{h}\left(\int_{\varepsilon-h}^{\varepsilon} |(\tau_\varepsilon - \mathrm{id})f(x+z)|\,\mathrm{d}z + \int_{2\varepsilon-h}^{2\varepsilon} |(\tau_\varepsilon - \mathrm{id})f(x+z)|\,\mathrm{d}z\right)\\ &= |(\tau_\varepsilon - \mathrm{id})f(x+\cdot)| * \rho_h(\varepsilon - h/2) + |(\tau_\varepsilon - \mathrm{id})f(x+\cdot)| * \rho_h(2\varepsilon - h/2)\\ &= |(\tau_\varepsilon - \mathrm{id})f| * \rho_h(x+\varepsilon - h/2) + |(\tau_\varepsilon - \mathrm{id})f| * \rho_h(x+2\varepsilon - h/2).\end{aligned}$$

Since the L^p norm on $\mathbb{T}$ is independent of translations, we obtain from (2.165) and (2.166) that

$$\begin{aligned}\|(\tau_h - \mathrm{id})\,\delta_\varepsilon f\|_{\mathrm{L}^p} \leqslant 2\,\frac{h}{2\varepsilon}\,\||(\tau_\varepsilon - \mathrm{id})f| * \rho_h\|_{\mathrm{L}^p} &\leqslant \frac{h}{\varepsilon}\|(\tau_\varepsilon - \mathrm{id})f\|_{\mathrm{L}^p}\,\|\rho_h\|_{\mathrm{L}^1}\\ &= \frac{h}{\varepsilon}\|(\tau_\varepsilon - \mathrm{id})f\|_{\mathrm{L}^p}.\end{aligned}$$ □

The same argument leads to the following n-dimensional version of Lemma 2.34.

Lemma 2.35. — *Let $f \in \mathrm{L}^p(\mathbb{T}^n)$ and let δ_ε the operator given by* (2.164)*. Then, for every $i \in [\![1,n]\!]$,*

$$\|(\tau_{he_i} - \mathrm{id})\,\delta_\varepsilon f\|_{\mathrm{L}^p(\mathbb{T}^n)} \leqslant \frac{h}{\varepsilon}\,\|(\tau_{\varepsilon e_i} - \mathrm{id})\,f\|_{\mathrm{L}^p(\mathbb{T}^n)}, \quad \forall\, h \in [0,\varepsilon].$$

2.7.7.3. A lemma on cuts. — The following lemma is due to Merlet [**38**], and was used in the proof of Proposition 2.6. We reproduce the argument in [**38**].

Lemma 2.36. — *Let $\alpha \in \mathbb{S}^1$. For every $z \in \mathbb{S}^1$, we let $\theta_\alpha(z)$ be the unique $\theta \in (\alpha - 2\pi, \alpha]$ such that $z = e^{\imath\theta}$. Then, for every $z, w \in \mathbb{S}^1$,*

$$\int_{\mathbb{S}^1} |\theta_\alpha(w) - \theta_\alpha(z)|\,\mathrm{d}\alpha = 2|\widehat{zw}|(2\pi - |\widehat{zw}|) \leqslant 4\pi|z-w|.$$

Here, $\widehat{zw}$ is (one of) the geodesic arc(s) that connects z and w on the circle, and $|\widehat{zw}|$ is the geodesic distance on the circle.

Proof. — It is easy to see that

$$|\theta_\alpha(z) - \theta_\alpha(w)| = \begin{cases} 2\pi - |\widehat{zw}|, & \text{if} \quad \alpha \in \widehat{zw}\\ |\widehat{zw}|, & \phantom{\text{if}} \quad \alpha \notin \widehat{zw}\end{cases}.$$

Hence

$$(2.167)\quad \int_{\mathbb{S}^1} |\theta_\alpha(w) - \theta_\alpha(z)|\,\mathrm{d}\alpha = \int_{\alpha\notin\widehat{zw}} |\widehat{zw}|\,\mathrm{d}\alpha + \int_{\alpha\in\widehat{zw}} (2\pi - |\widehat{zw}|)\,\mathrm{d}\alpha = 2|\widehat{zw}|(2\pi - |\widehat{zw}|).$$

We now use the inequality $\sin x \geqslant x(1 - x/\pi)$, valid for every $x \in [0, \pi/2]$, to find that

$$|z - w| = 2\left|\sin\frac{\widehat{zw}}{2}\right| \geqslant 2\frac{|\widehat{zw}|}{2}\left(1 - \frac{|\widehat{zw}|}{2\pi}\right) = \frac{1}{4\pi}2|\widehat{zw}|(2\pi - |\widehat{zw}|),$$

which together with (2.167) proves the lemma. □

2.7.7.4. Toolbox for the proof of Theorem 2.5. — We gather here the auxiliary results used in the proof of Theorem 2.5 in Subsection 2.7.3, as well as the proof of Lemma 2.9.

We start by establishing estimate (2.53), that we recall in the next statement.

Lemma 2.37. — *Let* $a, \varepsilon \in (0,1)$*. Then*

$$(1-a)^{\varepsilon} + a^{\varepsilon} - 1 \geqslant (1-\varepsilon)\, a^{\varepsilon}(1-a)^{\varepsilon}.$$

Proof. — By symmetry, we may assume that $a \leqslant 1/2$. By the mean value theorem, we have, for some $\xi \in (0,a)$,

$$1-(1-a)^{\varepsilon} = \varepsilon a(1-\xi)^{(\varepsilon-1)} \leqslant \varepsilon a^{\varepsilon}$$

$\big($since $1-\xi \geqslant a$ and therefore $(1-\xi)^{\varepsilon-1} \leqslant a^{\varepsilon-1}\big)$. Thus

$$(1-a)^{\varepsilon} + a^{\varepsilon} - 1 \geqslant (1-\varepsilon)\, a^{\varepsilon} \geqslant (1-\varepsilon)\, a^{\varepsilon}(1-a)^{\varepsilon}. \qquad \square$$

We continue with a proof of the estimate (2.55); this is the purpose of the next lemma.

Lemma 2.38. — *Let* $1<p<\infty$*. Set* $\delta(\varepsilon) \coloneqq \dfrac{1}{e^{1/\varepsilon}}$*, for every* $0<\varepsilon<1$*, and* $u_{\varepsilon} \coloneqq e^{i\varphi_{\delta(\varepsilon)}}$*, where* φ_{δ} *is given by* (2.54)*. Then* $|u_{\varepsilon}|_{W^{(1-\varepsilon)/p,p}} \approx 1$*.*

Proof. — We start with the following obvious estimate of $|u_{\varepsilon}|_{W^{(1-\varepsilon)/p,p}}$:

$$\begin{aligned}
|u_{\varepsilon}|^p_{W^{(1-\varepsilon)/p,p}} \approx & \int_0^{1/2}\int_{1/2}^{1/2+\delta} \frac{|e^{i2\pi(x-1/2)/\delta}-1|^p}{(x-y)^{2-\varepsilon}}\,dxdy \\
&+ \int_{1/2}^{1/2+\delta}\int_{1/2}^{1/2+\delta} \frac{|e^{i2\pi(x-1/2)/\delta}-e^{i2\pi(y-1/2)/\delta}|^p}{|x-y|^{2-\varepsilon}}\,dxdy \\
&+ \int_{1/2+\delta}^{1}\int_{1/2}^{1/2+\delta} \frac{|e^{i2\pi(x-1/2)/\delta}-1|^p}{(y-x)^{2-\varepsilon}}\,dxdy \eqqcolon I_1+I_2+I_3.
\end{aligned}$$

We next estimate each of the three integrals I_1, I_2 and I_3, using simple calculations and the fact that $\delta^{\varepsilon} = \dfrac{1}{e}$. To start with, we have

$$\begin{aligned}
I_2 &\approx \int_{1/2}^{1/2+\delta}\int_{1/2}^{1/2+\delta} \frac{\left|\sin\left(\dfrac{\pi(x-y)}{\delta}\right)\right|^p}{|x-y|^{2-\varepsilon}}\,dxdy \\
&= \left(\frac{\delta}{\pi}\right)^{\varepsilon}\int_{\pi/(2\delta)}^{\pi/(2\delta)+\pi}\int_{\pi/(2\delta)}^{\pi/(2\delta)+\pi} \frac{|\sin(x-y)|^p}{|x-y|^{2-\varepsilon}}\,dxdy \\
&= \left(\frac{\delta}{\pi}\right)^{\varepsilon}\int_{\pi/(2\delta)}^{\pi/(2\delta)+\pi}\left(\int_{\pi/(2\delta)-y}^{\pi/(2\delta)+\pi-y}\frac{|\sin t|^p}{|t|^{2-\varepsilon}}\,dt\right)dy \approx \int_{-\pi}^{\pi}\frac{|\sin t|^p}{|t|^{2-\varepsilon}}\,dt \approx 1;
\end{aligned}$$

the latter conclusion uses the fact that $p>1$.

We next estimate I_1 as follows.

$$
\begin{aligned}
I_1 &\approx \int_0^{1/2}\int_{1/2}^{1/2+\delta} \frac{\left|\sin\left(\dfrac{\pi(x-1/2)}{\delta}\right)\right|^p}{(x-y)^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y \approx \int_{-\pi/(2\delta)}^{0}\int_0^{\pi} \frac{\sin^p x}{(x-y)^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y\\
&= \int_0^{\pi}\int_x^{x+\pi/(2\delta)} \frac{\sin^p x}{t^{2-\varepsilon}}\,\mathrm{d}t\mathrm{d}x \approx \int_0^{\pi} \sin^p x\left(\frac{1}{x^{1-\varepsilon}} - \frac{1}{\left(x+\pi/(2\delta)\right)^{1-\varepsilon}}\right)\mathrm{d}x\\
&= \int_0^{\pi}\frac{\sin^p x}{x}\,\mathrm{d}x + o_\varepsilon(1) - \int_0^{\pi}\frac{\sin^p x}{\left(x+\pi/(2\delta)\right)^{1-\varepsilon}}\,\mathrm{d}x\\
&= \int_0^{\pi}\frac{\sin^p x}{x}\,\mathrm{d}x + o_\varepsilon(1) + O(\delta) = \int_0^{\pi}\frac{\sin^p x}{x}\,\mathrm{d}x + o_\varepsilon(1) \approx 1.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_3 &\approx \int_{1/2+\delta}^{1}\int_{1/2}^{1/2+\delta} \frac{\left|\sin\left(\dfrac{\pi(x-1/2)}{\delta}\right)\right|^p}{(y-x)^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y \approx \int_{\pi}^{\pi/(2\delta)}\int_0^{\pi} \frac{\sin^p x}{(y-x)^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y\\
&= \int_0^{\pi}\int_{\pi-x}^{\pi/(2\delta)-x} \frac{\sin^p x}{t^{2-\varepsilon}}\,\mathrm{d}t\mathrm{d}x \approx \int_0^{\pi} \sin^p x\left(\frac{1}{(\pi-x)^{1-\varepsilon}} - \frac{1}{\left(\pi/(2\delta)-x\right)^{1-\varepsilon}}\right)\mathrm{d}x \approx 1.
\end{aligned}
$$

By the above estimates of I_1, I_2 and I_3, we conclude that $|u_\varepsilon|_{\mathrm{W}^{(1-\varepsilon)/p,p}} \approx 1$ as $\varepsilon \to 0$. □

We next present a proof the Lemma 2.9, in the spirit of [**11**, Lemma 2].

Proof of Lemma 2.9. — By scale invariance, we may assume that $I = (0,1)$.

For every $\ell \in \mathbb{Z}$, we define the sets $A_\ell \coloneqq \{x \in I \mid \psi(x) < \ell\}$. Since (A_ℓ) is a non-decreasing sequence with $|A_\ell| \to 0$ when $\ell \to -\infty$ and $|A_\ell| \to 1$ when $\ell \to \infty$, there exists some $k \in \mathbb{Z}$ such that $|A_k| \leqslant \dfrac{1}{2}$ and $|A_{k+1}| > \dfrac{1}{2}$.

Note that

$$
(2.168)\quad |\psi(x)-\psi(y)| \geqslant 1, \qquad \forall\, \ell \in \mathbb{Z}, \forall\, x \in A_\ell, \forall\, y \in {}^c(A_\ell).
$$

Hence, by applying inequality (2.51) first to A_k and next to A_{k+1}, and by using (2.168), we get

$$
\frac{1}{2}\,|A_k| \leqslant \left(C\varepsilon\int_{A_k}\int_{^c(A_k)} \frac{1}{|x-y|^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y\right)^{1/\varepsilon} \leqslant \left(C\varepsilon\int_{A_k}\int_{^c(A_k)} \frac{|\psi(x)-\psi(y)|}{|x-y|^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y\right)^{1/\varepsilon}
$$

and

$$
\begin{aligned}
\frac{1}{2}\,|{}^c(A_{k+1})| &\leqslant \left(C\varepsilon\int_{A_{k+1}}\int_{^c(A_{k+1})} \frac{1}{|x-y|^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y\right)^{1/\varepsilon}\\
&\leqslant \left(C\varepsilon\int_{A_{k+1}}\int_{^c(A_{k+1})} \frac{|\psi(x)-\psi(y)|}{|x-y|^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y\right)^{1/\varepsilon}.
\end{aligned}
$$

We find that:

$$
|\{x \in I \mid \psi(x) \neq k\}| = |A_k| + |{}^c(A_{k+1})| \leqslant 4\left(C\varepsilon\int_I\int_I \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{2-\varepsilon}}\,\mathrm{d}x\mathrm{d}y\right)^{1/\varepsilon}. \qquad \square
$$

Lemma 2.39. — *Let* φ *be a lifting of* $u = e^{\imath\varphi_\delta}$*, where* φ_δ *is given by* (2.54)*, i.e.,*

$$\varphi_\delta(x) \coloneqq \begin{cases} 0, & x < \dfrac{1}{2} \\ \dfrac{(2x-1)\pi}{\delta}, \text{ if } & \dfrac{1}{2} < x < \dfrac{1}{2} + \delta \\ 2\pi, & \dfrac{1}{2} + \delta < x \end{cases} .$$

Let $\psi \coloneqq \dfrac{\varphi - \varphi_\delta}{2\pi}$.

Then, if $x, y \in \left(0, \dfrac{1}{2} + \dfrac{2\delta}{3}\right)$*, or if* $x, y \in \left(\dfrac{1}{2} + \dfrac{\delta}{3}, 1\right)$*, we have*

$$\text{(2.169)} \quad |\psi(x) - \psi(y)| \leqslant |\varphi(x) - \varphi(y)|.$$

Proof. — We will verify (2.169) when $x, y \in \left(0, \dfrac{1}{2} + \dfrac{2\delta}{3}\right)$, since the proof when x and y both belong to the second interval is similar. Estimate (2.169) being clear when $0 < x \leqslant \dfrac{1}{2}$ and $0 < y \leqslant \dfrac{1}{2}$, we may assume that $y > \dfrac{1}{2}$.

To summarize, we will establish (2.169) when $y \in \left(\dfrac{1}{2}, \dfrac{1}{2} + \dfrac{2\delta}{3}\right)$ and $x \in \left(0, \dfrac{1}{2} + \dfrac{2\delta}{3}\right)$. Two cases will be considered: $x \in \left(0, \dfrac{1}{2}\right]$ and $x \in \left(\dfrac{1}{2}, \dfrac{1}{2} + \dfrac{2\delta}{3}\right)$.

Since φ and φ_δ are liftings of the same function u, for every x there exists an integer $k(x)$ such that $\varphi(x) = \varphi_\delta(x) + 2\pi k(x)$. Same for y. We may always assume, with no loss of generality, that $k(y) = 0$.

To start with, assume that $x \in \left(0, \dfrac{1}{2}\right]$. Then (2.169) is equivalent to

$$\text{(2.170)} \quad |k(x)| \leqslant \left| 2\pi k(x) - \frac{(2y-1)\pi}{\delta} \right| = |2\pi k(x) - Y|,$$

where we let $Y \coloneqq \dfrac{(2y-1)\pi}{\delta}$. Note that $0 < Y < \dfrac{4\pi}{3}$. If $k(x) \leqslant 0$, then (2.170) is obviously true. In the case where $k(x) > 0$ is nonnegative, (2.170) becomes

$$\text{(2.171)} \quad (2\pi - 1)k(x) \geqslant Y,$$

and follows from $Y < \dfrac{4\pi}{3}$.

Suppose next that we have $x \in \left(\dfrac{1}{2}, \dfrac{1}{2} + \dfrac{2\delta}{3}\right)$. Then (2.169) becomes

$$\text{(2.172)} \quad |k(x)| \leqslant \left| \frac{2(x-y)\pi}{\delta} + 2\pi k(x) \right| = |X + 2\pi k(x)|,$$

where $X \coloneqq \dfrac{2(x-y)\pi}{\delta}$. Note that $-\dfrac{4\pi}{3} < X < \dfrac{4\pi}{3}$. We investigate the validity of (2.172) when $X \geqslant 0$; the case where $X < 0$ is similar. When $X \geqslant 0$, inequality (2.172) is always true if $k(x)$ is non negative. When $k(x) < 0$, (2.172) amounts to

$$(2\pi - 1)(-k(x)) \geqslant X,$$

which holds since $X < \dfrac{4\pi}{3}$. □

Lemma 2.40. — *Let A, $B \subset (a,b)$ be such that "A is on the left of B", i.e., $y < x$, $\forall\, y \in A$, $\forall\, x \in B$. Define $A_\ell := (a, a+|A|)$ and $B_r := (b-|B|, b)$. Let $t > 0$.*

Then

$$\int_A \int_B \frac{1}{(x-y)^t}\,\mathrm{d}x\mathrm{d}y \geqslant \int_{A_\ell}\int_{B_r} \frac{1}{(x-y)^t}\,\mathrm{d}x\mathrm{d}y.$$

Proof. — It suffices to establish the inequality

$$\int_A \frac{\mathrm{d}y}{(x-y)^t} \geqslant \int_{A_\ell} \frac{\mathrm{d}y}{(x-y)^t}, \quad \forall\, x \in (a,b) \text{ such that } y < x, \forall\, y \in A. \tag{2.173}$$

Indeed, assume that (2.173) holds. Then by symmetry we have

$$\int_B \frac{\mathrm{d}x}{(x-y)^t} \geqslant \int_{B_r} \frac{\mathrm{d}x}{(x-y)^t}, \quad \forall\, y \in (a,b) \text{ such that } x > y, \forall\, x \in B. \tag{2.174}$$

By (2.173) and (2.174), we have

$$\begin{aligned}\int_B \left(\int_A \frac{\mathrm{d}y}{(x-y)^t}\right)\mathrm{d}x \geqslant \int_B \left(\int_{A_\ell} \frac{\mathrm{d}y}{(x-y)^t}\right)\mathrm{d}x &= \int_{A_\ell} \left(\int_B \frac{\mathrm{d}x}{(x-y)^t}\right)\mathrm{d}y \\ &\geqslant \int_{A_\ell}\int_{B_r} \frac{1}{(x-y)^t}\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

It remains to prove (2.173). We first note that (2.173) is true when A is an interval (by explicit calculation of both sides in (2.173)). By a standard argument, we find that (2.173) holds: first when A is an open set, next when A is compact, and finally for every measurable A. □

Lemma 2.41. — *Let $0 < s < 1$ and $1 \leqslant p < \infty$. Then, for any $\varphi \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{R})$, we have*

$$|\varphi|^p_{\mathrm{W}^{s,p}(\mathbb{T}^n)} \gtrsim \int_{\mathbb{T}^{n-1}} |\varphi(\cdot, x_2, \ldots, x_n)|^p_{\mathrm{W}^{s,p}(\mathbb{T})}\,\mathrm{d}x_2 \ldots \mathrm{d}x_n. \tag{2.175}$$

Proof. — Let A denote the integral in the right-hand side of (2.175), that is,

$$A = \int_{\mathbb{T}^{n-1}}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|\varphi(x_1,x') - \varphi(z_1,x')|^p}{|x_1 - z_1|^{1+sp}}\,\mathrm{d}x_1\mathrm{d}z_1\mathrm{d}x'.$$

We will use the notation $x := (x_1, x')$ and $z := (z_1, x')$, with $x_1, z_1 \in \mathbb{T}$ and $x' \in \mathbb{T}^{n-1}$. Integrating the inequality

$$|\varphi(x) - \varphi(z)|^p \leqslant 2^{p-1}\big(|\varphi(x)-\varphi(y)|^p + |\varphi(y) - \varphi(z)|^p\big), \quad \forall\, y \in \mathbb{T}^n,$$

with respect to $y \in B\left(\dfrac{x+z}{2}, \dfrac{|x_1 - z_1|}{4}\right)$, we find that

$$A \lesssim \int_{\mathbb{T}^n}\int_{\mathbb{T}} \fint_{B((x+z)/2, |x_1-z_1|/4)} \frac{|\varphi(x)-\varphi(y)|^p}{|x_1-z_1|^{1+sp}}\,\mathrm{d}y\mathrm{d}z_1\mathrm{d}x.$$

Noting that $B\left(\dfrac{x+z}{2}, \dfrac{|x_1-z_1|}{4}\right) \subset B\left(x, \dfrac{3|x_1-z_1|}{4}\right)$, we find that

$$\begin{aligned} A &\lesssim \int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\int_{|x_1-z_1|\geqslant 4|x-y|/3} \frac{dz_1}{|x_1-z_1|^{n+1+sp}} |\varphi(x)-\varphi(y)|^p\,\mathrm{d}y\mathrm{d}x \\ &\lesssim \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\mathrm{d}x. \end{aligned}$$

□

Lemma 2.42. — *Let* $0 < s < 1$ *and* $1 \leqslant p < \infty$. *Set*

$$K^n(x_1, y_1) \coloneqq \int_{\mathbb{T}^{n-1}} \int_{\mathbb{T}^{n-1}} \frac{1}{|(x_1, x') - (y_1, y')|^{n+sp}} \, \mathrm{d}x' \mathrm{d}y', \quad \forall\, x_1, y_1 \in \mathbb{T}.$$

Then we have

$$K^n(x_1, y_1) \lesssim \frac{1}{|x_1 - y_1|^{1+sp}}.$$

Proof. — Set $t \coloneqq x_1 - y_1$ and $z' \coloneqq x' - y'$. Then

$$\begin{aligned} K^n(x_1, y_1) &= \int_{\mathbb{T}^{n-1}} \int_{\mathbb{T}^{n-1}} \frac{\mathrm{d}x' \mathrm{d}y'}{|(t, x' - y')|^{n+sp}} \leqslant \int_{|x'| \leqslant 1} \int_{|z'| \leqslant 2} \frac{\mathrm{d}z' \mathrm{d}x'}{|(t, z')|^{n+sp}} \\ &\lesssim \int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}z'}{|(t, z')|^{n+sp}} \lesssim \frac{1}{|t|^{1+sp}}. \end{aligned}$$

□

Corollary 2.43. — *Let* $s > 0$ *and* $1 \leqslant p < \infty$. *Let* $f \in \mathrm{W}^{s,p}(\mathbb{T}; \mathbb{C})$ *and consider the function* $F\colon \mathbb{T}^n \to \mathbb{C}$ *defined by* $F(x_1, x') \coloneqq f(x_1)$, $\forall\, x = (x_1, x') \in \mathbb{T}^n$.

Then $F \in \mathrm{W}^{s,p}(\mathbb{T}^n; \mathbb{C})$ *and* $|F|_{\mathrm{W}^{s,p}(\mathbb{T}^n)} \approx |f|_{\mathrm{W}^{s,p}(\mathbb{T})}$.

Proof. — If k is an integer and $k \leqslant s$, then we clearly have

$$\|\mathrm{D}^k F\|_{\mathrm{L}^p(\mathbb{T}^n)} = \|\mathrm{D}^k f\|_{\mathrm{L}^p(\mathbb{T})}. \tag{2.176}$$

In particular, the conclusion of the lemma holds when s is an integer.

Suppose now that s is not an integer and write $s = [s] + \sigma$, with $\sigma \in (0, 1)$. By Lemma 2.41, we have

$$\begin{aligned} |\mathrm{D}^{[s]} F|^p_{\mathrm{W}^{\sigma,p}(\mathbb{T}^n)} &\gtrsim \int_{\mathbb{T}^{n-1}} |\mathrm{D}^{[s]} F(\cdot, x')|^p_{\mathrm{W}^{\sigma,p}(\mathbb{T})} \, \mathrm{d}x' = \int_{\mathbb{T}^{n-1}} |\mathrm{D}^{[s]} f|^p_{\mathrm{W}^{\sigma,p}(\mathbb{T})} \, \mathrm{d}x' \\ &= |\mathrm{D}^{[s]} f|^p_{\mathrm{W}^{\sigma,p}(\mathbb{T})}. \end{aligned} \tag{2.177}$$

On the other hand, using Lemma 2.42 for $s = \sigma$, we obtain

$$\begin{aligned} |\mathrm{D}^{[s]} F|^p_{\mathrm{W}^{\sigma,p}(\mathbb{T}^n)} &= \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{|\mathrm{D}^{[s]} F(x_1, x') - \mathrm{D}^{[s]} F(y_1, y')|^p}{|x - y|^{n+\sigma p}} \, \mathrm{d}x \mathrm{d}y \\ &= \int_0^1 \int_0^1 |\mathrm{D}^{[s]} f(x_1) - \mathrm{D}^{[s]} f(y_1)|^p K^n(x_1, y_1) \, \mathrm{d}x_1 \mathrm{d}x_2 \\ &\lesssim \int_0^1 \int_0^1 \frac{|\mathrm{D}^{[s]} f(x_1) - \mathrm{D}^{[s]} f(y_1)|^p}{|x_1 - y_1|^{1+\sigma p}} \, \mathrm{d}x_1 \mathrm{d}x_2 = |\mathrm{D}^{[s]} f|^p_{\mathrm{W}^{\sigma,p}(\mathbb{T})}. \end{aligned} \tag{2.178}$$

From (2.176), (2.177) and (2.178), we have $|F|_{\mathrm{W}^{s,p}(\mathbb{T}^n)} \approx |f|_{\mathrm{W}^{s,p}(\mathbb{T})}$. □

2.7.7.5. Toolbox for optimal estimates when $sp \geqslant 1$. — In this subsection, we establish the auxiliary results required in Subsection 2.7.4.

Lemma 2.44. — *Let* $s > 0$ *and* $1 \leqslant p < \infty$. *Let* $f \in \mathrm{C}^\infty_c((0, 1); \mathbb{C})$, $f \not\equiv 0$. *Consider the functions* $f_j \coloneqq \sum_{0 \leqslant k \leqslant j-1} f(xj - k)$, $\forall\, j \geqslant 1$. *Then*

$$|f_j|_{\mathrm{W}^{s,p}((0,1))} \approx j^s. \tag{2.179}$$

Proof. — If s, k are integers and $k \leqslant s$, then we have

$$\|\mathrm{D}^k f_j\| \approx j^k. \tag{2.180}$$

In particular, (2.179) holds when s is an integer.

Suppose now that $s \notin \mathbb{N}$ and let $\sigma \coloneqq s - [s] \in (0,1)$. Then we have

$$|\mathrm{D}^{[s]} f_j|^p_{\mathrm{W}^{\sigma,p}} = j^{[s]p} \sum_{k,\ell=0}^{j-1} I_{k,\ell}, \tag{2.181}$$

with

$$\begin{aligned} I_{k,\ell} &\coloneqq \int_{k/j}^{(k+1)/j} \int_{\ell/j}^{(\ell+1)/j} \frac{|f^{([s])}(xj-l) - f^{([s])}(yj-k)|^p}{|x-y|^{1+\sigma p}} \,\mathrm{d}x\mathrm{d}y \\ &= j^{1-\sigma p} \int_0^1 \int_0^1 \frac{|f^{([s])}(X) - f^{([s])}(Y)|^p}{|\ell - k + X - Y|^{1+\sigma p}} \,\mathrm{d}X\mathrm{d}Y. \end{aligned}$$

If $k \neq \ell$, then $I_{k,\ell}$ can be estimated as follows.

$$I_{k,\ell} \lesssim \sup |f^{([s])}|^p \frac{j^{\sigma p-1}}{|\ell - k|^{1+\sigma p}}. \tag{2.182}$$

(When $|\ell - k| \geqslant 2$, estimate (2.182) follows from the fact that $|\ell - k + X - Y| \approx |\ell - k|$. When $|\ell - k| = 1$, we rely on the fact that $f \in \mathrm{C}^\infty_c((0,1))$, and thus there exists some $\varepsilon > 0$ such that $|f^{([s])}(X) - f^{([s])}(Y)| = 0$ when $|\ell - k + X - Y| \leqslant \varepsilon$.)

Thus

$$\sum_{k \neq \ell} I_{k,\ell} \lesssim j^{\sigma p-1} \sum_{\ell=1}^{j-1} \sum_{k=0}^{\ell-1} \frac{1}{(l-k)^{1+\sigma p}} \lesssim j^{\sigma p-1} \left(j \sum_{k=1}^{j-1} \frac{1}{k^{1+\sigma p}} - \sum_{k=1}^{j-1} \frac{1}{k^{\sigma p}} \right) \lesssim j^{\sigma p}. \tag{2.183}$$

On the other hand, for $k = \ell$ we obtain

$$\begin{aligned} I_{k,k} &= \int_{k/j}^{(k+1)/j} \int_{k/j}^{(k+1)/j} \frac{|f^{([s])}(xj-k) - f^{([s])}(yj-k)|^p}{|x-y|^{1+\sigma p}} \,\mathrm{d}x\mathrm{d}y \\ &= j^{\sigma p-1} |f^{([s])}|^p_{\mathrm{W}^{s,p}((0,1))} \approx j^{\sigma p-1}. \end{aligned}$$

Therefore, we have

$$\sum_{k=0}^{j-1} I_{k,k} \approx j^{\sigma p}. \tag{2.184}$$

By combining (2.180) – (2.184), we find that $|\mathrm{D}^{[s]} f_j|_{\mathrm{W}^{\sigma,p}((0,1))} \approx j^s$, and therefore $|f_j|_{\mathrm{W}^{s,p}((0,1))} \approx j^s$. □

The next result is a variant of [**10**, Lemma D.2].

Lemma 2.45. — *Let $s \geqslant 1$, $1 \leq p < \infty$ and $v \in \mathrm{W}^{s,p}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\mathbb{R}^n)$. Then*

$$\|v \wedge \nabla v\|_{\mathrm{W}^{s-1,p}} \lesssim \|v\|_{\mathrm{L}^\infty} \|v\|_{\mathrm{W}^{s,p}}. \tag{2.185}$$

The proof of Lemma 2.45, as well as the one of Lemma 2.56, relies on Littlewood-Paley decompositions. We gather here some standard properties of such decompositions.

Let $\zeta \in \mathrm{C}^\infty_c(B(0,1); \mathbb{R}_+)$ be such that

$$\zeta \equiv 1 \text{ in } \overline{B}\left(0, \frac{3}{4}\right) \quad \text{and} \quad \operatorname{supp} \zeta \subset B\left(0, \frac{4}{5}\right). \tag{2.186}$$

Define φ_j, $j \geqslant 0$, by

$$\widehat{\varphi_0}(\xi) := \zeta(\xi) \text{ and, for every } j \geqslant 1,\ \widehat{\varphi_j}(\xi) := \zeta\left(\xi/2^{j+1}\right) - \zeta\left(\xi/2^j\right). \tag{2.187}$$

Given $f \in \mathcal{S}'$, we let $f = \sum f_j = \sum f * \varphi_j$ be its Littlewood-Paley decomposition, and recall ([**53**, Section 2.3.1, Definition 2, p. 45], [**53**, Section 2.5.7, Theorem p. 90]) that

$$\|f\|_{B^0_{\infty,\infty}} \sim \sup_j \|f_j\|_{\mathrm{L}^\infty}, \tag{2.188}$$

$$\|f\|^p_{\mathrm{W}^{s,p}} \sim \sum_j 2^{spj}\|f_j\|^p_{\mathrm{L}^p}, \quad \forall\, s > 0, \forall\, 1 \leqslant p < \infty,\ s \text{ non integer}. \tag{2.189}$$

Recall also the following Nikolskiĭ type inequalities [**55**]. Set $\mathcal{C}_0 := B(0,2)$ and, for $j \geqslant 1$, $\mathcal{C}_j := B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$. If $f^j \in \mathcal{S}'$ and

$$\operatorname{supp} \widehat{f^j} \subset \bigcup_{|\ell - j| \leqslant k} \mathcal{C}_\ell \text{ for some fixed } k, \tag{2.190}$$

then

$$\Big\|\sum_j f^j\Big\|_{B^0_{\infty,\infty}} \lesssim \sup_j \|f^j\|_{\mathrm{L}^\infty} \tag{2.191}$$

and

$$\Big\|\sum_j f^j\Big\|^p_{\mathrm{W}^{s,p}} \lesssim \sum_j 2^{spj}\|f^j\|^p_{\mathrm{L}^p}, \quad \forall\, s > 0, \forall\, 1 \leqslant p < \infty,\ s \text{ non integer}. \tag{2.192}$$

Remark 2.46. — *The inequality* (2.192) *also holds if the assumption* (2.190) *is weakened to* $\operatorname{supp} \widehat{f^j} \subset B(0, 2^{j+k})$ *for some fixed* k *(*[**18**, Lemma 1]*; see also* [**55**]*).*

We next recall the following standard inequalities; see e.g. [**22**, Lemma 2.1.1] for the first result, and [**18**, Corollary 1, Lemma 2] for the next one.

Lemma 2.47. — *Let* $f \in \mathcal{S}'(\mathbb{R}^n)$ *be such that* $\operatorname{supp} \widehat{f} \subset B(0,R)$*. Then, for any* $1 \leqslant q \leqslant \infty$*,*

$$\|\nabla f\|_{\mathrm{L}^q} \leqslant C(q) R \|f\|_{\mathrm{L}^q}.$$

Lemma 2.48. — *Let* $1 \leqslant q \leqslant \infty$ *and* $f \in \mathrm{L}^q(\mathbb{R}^n)$*. Let* $f = \sum f_i$ *be the Littlewood-Paley decomposition of* f*. Then*

$$\Big\|\sum_{k \leqslant j} f_k\Big\|_{\mathrm{L}^q} \leqslant C(q)\|f\|_{\mathrm{L}^q}.$$

Proof of Lemma 2.45. — Suppose first that $s \geqslant 1$ is an integer. Then we have

$$\begin{aligned} \|v \wedge \nabla v\|^p_{\mathrm{W}^{s-1,p}} &\lesssim \|v \wedge \nabla v\|^p_{\mathrm{L}^p} + \left\|\mathrm{D}^{s-1}(v \wedge \nabla v)\right\|^p_{\mathrm{L}^p} \\ &\lesssim \|v\|^p_{\mathrm{L}^\infty}\|\nabla v\|^p_{\mathrm{L}^p} + \sum_{|\alpha|+|\beta|=s} \left\|\mathrm{D}^\alpha v \wedge \mathrm{D}^\beta v\right\|^p_{\mathrm{L}^p}. \end{aligned} \tag{2.193}$$

By applying the Hölder and the Gagliardo-Nirenberg inequalities, we find that, for every m_1, $m_2 \in \mathbb{N}$ with $m_1 + m_2 = s$,

$$\begin{aligned} \|\mathrm{D}^{m_1} v \wedge \mathrm{D}^{m_2} v\|_{\mathrm{L}^p} &\lesssim \|\mathrm{D}^{m_1} v\|_{\mathrm{L}^{sp/m_1}} \|\mathrm{D}^{m_2} v\|_{\mathrm{L}^{sp/m_2}} \\ &\lesssim \left(\|v\|^{1-m_1/s}_{\mathrm{L}^\infty} \|\mathrm{D}^s v\|^{m_1/s}_{\mathrm{L}^p}\right)\left(\|v\|^{1-m_2/s}_{\mathrm{L}^\infty} \|\mathrm{D}^s v\|^{m_2/s}_{\mathrm{L}^p}\right) \\ &= \|v\|_{\mathrm{L}^\infty}\|\mathrm{D}^s v\|_{\mathrm{L}^p} \lesssim \|v\|_{\mathrm{L}^\infty}\|v\|_{\mathrm{W}^{s,p}}, \end{aligned}$$

which, together with (2.193), proves (2.185).

We next assume that $s > 1$ is not an integer. In this case, the proof uses the same idea as in [**10**, Lemma D.2]. We consider the Littlewood-Paley decomposition of v in $\mathcal{S}'(\mathbb{R}^n)$, $v = \sum_{j\geqslant 0} v_j \coloneqq \sum_{j\geqslant 0} v * \varphi_j$, with the functions φ_j previously defined by (2.186) and (2.187).

We next define

$$r_j \coloneqq v_j \wedge \nabla \sum_{k<j} v_k,\ \forall\, j \geqslant 1,\ r_0 \coloneqq 0,\ \text{and}\ s_j \coloneqq \sum_{k\leqslant j} v_k \wedge \nabla v_j,\ \forall\, j \geqslant 0.$$

Then we have $v \wedge \nabla v = \sum_{j\geqslant 0}(r_j + s_j)$. Note that $\operatorname{supp}(\widehat{r_j} + \widehat{s_j}) \subset B(0, 2^{j+2})$ and that $s-1 > 0$. Hence, by (2.192) and Remark 2.46, we have that

$$\begin{aligned}\|v\wedge\nabla v\|^p_{\mathrm{W}^{s-1,p}} &= \Big\|\sum_{j\geqslant 0}(r_j+s_j)\Big\|^p_{\mathrm{W}^{s-1,p}} \lesssim \sum_{j\geqslant 0} 2^{(s-1)pj}\|r_j+s_j\|^p_{\mathrm{L}^p}\\ &\lesssim \sum_{j\geqslant 0} 2^{(s-1)pj}\left(\|r_j\|^p_{\mathrm{L}^p} + \|s_j\|^p_{\mathrm{L}^p}\right).\end{aligned}\tag{2.194}$$

We will now estimate $\|r_j\|_{\mathrm{L}^p}$ and $\|s_j\|_{\mathrm{L}^p}$ using Lemmas 2.47 and 2.48. First, since $\operatorname{supp}\sum_{k<j}\widehat{v_k} \subset B(0,2^{j+1})$, we have

$$\|r_j\|_{\mathrm{L}^p} \leqslant \|v_j\|_{\mathrm{L}^p}\Big\|\nabla\sum_{k<j} v_k\Big\|_{\mathrm{L}^\infty} \lesssim 2^j\|v_j\|_{\mathrm{L}^p}\Big\|\sum_{k<j} v_k\Big\|_{\mathrm{L}^\infty} \lesssim 2^j\|v_j\|_{\mathrm{L}^p}\|v\|_{\mathrm{L}^\infty}.$$

Next, since $\operatorname{supp}\widehat{v_j} \subset B(0,2^{j+1})$, we have

$$\|s_j\|_{\mathrm{L}^p} \leqslant \Big\|\sum_{k\leqslant j} v_k\Big\|_{\mathrm{L}^\infty}\|\nabla v_j\|_{\mathrm{L}^p} \lesssim 2^j\Big\|\sum_{k\leqslant j} v_k\Big\|_{\mathrm{L}^\infty}\|v_j\|_{\mathrm{L}^p} \lesssim 2^j\|v\|_{\mathrm{L}^\infty}\|v_j\|_{\mathrm{L}^p}.$$

Combining the two above estimates with (2.189), (2.192) and (2.194), we find

$$\|v\wedge\nabla v\|^p_{\mathrm{W}^{s-1,p}} \lesssim \|v\|^p_{\mathrm{L}^\infty}\sum_{j\geqslant 0}2^{spj}\|v_j\|^p_{\mathrm{L}^p} \lesssim \|v\|^p_{\mathrm{L}^\infty}\|v\|^p_{\mathrm{W}^{s,p}}.$$ □

We now turn to the the proof of some estimates used in the different proofs of Theorem 2.12 (Lemmas 2.49-2.55).

The next result appears in Merlet [**38**]. We present below a simplified argument.

Lemma 2.49. — *Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp > 1$, and let $0 \leqslant x \leqslant y \leqslant 1$. Let $u \in \mathrm{W}^{s,p}(\mathbb{T};\mathbb{S}^1)$ and let $\varphi \in \mathrm{W}^{s,p}(\mathbb{T};\mathbb{R})$ be a lifting of u. Then*

$$|\varphi(x)-\varphi(y)|^p \lesssim |u(x)-u(y)|^p + (y-x)^{p-1/s}|u|^{p/s}_{\mathrm{W}^{s,p}((x,y))}.$$

Proof. — We will show that

(a) $|\varphi(x)-\varphi(y)| \leqslant \pi \Longrightarrow |\varphi(x)-\varphi(y)| \lesssim |u(x)-u(y)|$.

(b) $|\varphi(x)-\varphi(y)| > \pi \Longrightarrow |\varphi(x)-\varphi(y)| \lesssim (y-x)^{1-1/sp}|u|^{1/s}_{\mathrm{W}^{s,p}((x,y))}$.

The first case is obvious. Indeed, if $|\varphi(x)-\varphi(y)| \leqslant \pi$, then

$$|\varphi(x)-\varphi(y)| \leqslant \pi\left|\sin\frac{\varphi(x)-\varphi(y)}{2}\right| = \frac{\pi}{2}|u(x)-u(y)|.$$

Consider now the case where $|\varphi(x)-\varphi(y)| > \pi$. We may assume that $\varphi(x) = 0$. In addition, using the monotonicity in y of the right-hand side of (b), it suffices to establish

(b) when y is replaced by $z \in [x, y]$ such that $|\varphi(z)| = \max\limits_{[x,y]} |\varphi|$. Therefore, with no loss of generality we may assume that $\varphi(y) = \max\limits_{[x,y]} |\varphi|$. Let α be such that $\pi < \alpha < \min\{|\varphi(y)|, 2\pi\}$, and decompose the interval $[x, y]$ as

$$[x, y] = [x_0, x_1] \cup [x_1, x_2] \cup \ldots \cup [x_J, x_{J+1}],$$

with

$$\begin{aligned} &x_0 \coloneqq x,\ J \coloneqq \left[\frac{\varphi(y)}{\alpha}\right], \quad x_{J+1} \coloneqq y \\ &x_j \coloneqq \text{ the smallest solution } t \text{ of } \varphi(t) = j\alpha,\ \forall\, j \in [\![1, J]\!]. \end{aligned} \tag{2.195}$$

We then have $J \geqslant 1$ and, by (2.195),

$$|\varphi(x) - \varphi(y)| = \varphi(y) < \alpha(J+1) \lesssim J. \tag{2.196}$$

We next note, as in [**38**], the following quantitative form of the continuous embedding
$W^{s,p}((t,z)) \hookrightarrow C^{0,s-1/p}([t,z])$, with $sp > 1$ and $0 \leqslant t \leqslant z \leqslant 1$:

$$|u(z_0) - u(t_0)| \leqslant c\,(z-t)^{s-1/p} |u|_{W^{s,p}((z,t))}, \quad \forall\, 0 \leqslant t < t_0 < z_0 < z \leqslant 1, \tag{2.197}$$

where $c = c(s,p)$. Estimate (2.197) follows e.g. from [**29**, Lemma 1.1] (with $\Psi(t) \coloneqq |t|^p$ and $p(t) \coloneqq |t|^{s+1/p}$).

When $|\varphi(z) - \varphi(t)| > \pi$ for some $t < z$, (2.197) implies that we necessarily have

$$(z-t)^{s-1/p} |u|_{W^{s,p}((t,z))} > \frac{1}{c}$$

(with c as in (2.197)). Indeed, argue by contradiction. If $(z-t)^{s-1/p}|u|_{W^{s,p}((t,z))} \leqslant \dfrac{1}{c}$, then, by (2.197), we have

$$|u(z_0) - u(t_0)| \leqslant 1, \qquad \forall\ 0 \leqslant t < t_0 < z_0 < z \leqslant 1. \tag{2.198}$$

Using (2.198) and the continuity of u and φ, we find that $|\varphi(z_0) - \varphi(t_0)| \leqslant \dfrac{\pi}{3}$ for every t_0 and z_0 as above. In particular, we obtain the contradiction $\varphi(z) - \varphi(t) < \pi$.

Therefore, for every $1 \leqslant j \leqslant J$, we have

$$(x_j - x_{j-1})^{s-1/p}\, |u|_{W^{s,p}((x_{j-1},x_j))} > \frac{1}{c}. \tag{2.199}$$

Using (2.199), we find that

$$J = c^{1/s} \sum_{1 \leqslant j \leqslant J} \frac{1}{c^{1/s}} \leqslant c^{1/s} \sum_{1 \leqslant j \leqslant J} (x_j - x_{j-1})^{1-1/sp}\, |u|^{1/s}_{W^{s,p}((x_{j-1},x_j))}. \tag{2.200}$$

Applying in (2.200) Hölder's inequality with the exponents $\dfrac{sp}{sp-1}$ and sp, we obtain

$$J \lesssim \left(\sum_{1 \leqslant j \leqslant J} (x_j - x_{j-1}) \right)^{1-1/sp} \left(\sum_{1 \leqslant j \leqslant J} |u|^p_{W^{s,p}((x_{j-1},x_j))} \right)^{1/sp} \lesssim (y-x)^{1-1/sp}\, |u|^{1/s}_{W^{s,p}(x,y)}.$$

This combined with (2.196) proves the assertion (b). □

We next establish several estimates used in the third proof of Theorem 2.12.

We start with the proof of (2.87). This estimate is certainly well-known to experts, but we were unable to find it in the literature and we present an argument for the sake of completeness.

Lemma 2.50. — *Let $f \in \mathrm{L}^1(\mathbb{S}^1;\mathbb{C})$ and let $\widetilde{f}$ be the harmonic extension of f. Let Mf be the maximal function of f. Then we have*

$$|\widetilde{f}(r\omega)| \leqslant Mf(\omega), \quad \forall\, \omega \in \mathbb{S}^1, \quad \forall\, r \in [0,1].$$

Proof. — Let $P(x,y)$ be the Poisson kernel on the unit disc, i.e., $P(x,y) \coloneqq \dfrac{1-r^2}{2\pi|x-y|^2}$. Here, $x = r\omega$, $\omega \in \mathbb{S}^1$, $r \in [0,1]$, and $y \in \mathbb{S}^1$. We note that $P(x,\cdot)$ is positive and "symmetric with respect to $O\omega$ and decreasing in y". More specifically, if y and y' are symmetric with respect to $O\omega$, then $P(x,y) = P(x,y')$. On the other hand, $P(x,\cdot)$ decreases with the distance from y to ω. This allows us to mimic the proof in [**52**, Chapter II, Section 2.1, formula (17), p. 57] and obtain the estimate

$$|\widetilde{f}(x)| \leqslant \int_{\mathbb{S}^1} |f(y)|P(x,y)\,\mathrm{d}y \leqslant Mf(\omega)\int_{\mathbb{S}^1} P(x,y)\,\mathrm{d}y = Mf(\omega). \qquad \square$$

We continue with the proof of the estimate (2.90), that we restate here.

Lemma 2.51. — *Let $1 \leqslant p < \infty$ and $0 < s < 1$ be such that $sp \geqslant 1$. Let $u \in \mathrm{W}^{s,p}(\mathbb{S}^1;\mathbb{S}^1)$ and let $\widetilde{u}$ be given by* (2.89)*. Define $\varepsilon(\omega) \coloneqq \displaystyle\int_0^1 |\operatorname{Jac}\widetilde{u}(r\omega)|\,\mathrm{d}r$, $\forall\, \omega \in \mathbb{S}^1$. Then*

$$\|\varepsilon\|_{\mathrm{L}^{sp}} \lesssim |u|^{1/s}_{\mathrm{W}^{s,p}}. \tag{2.201}$$

In the proof of the above lemma, we will need the following cousin of [**13**, Lemma 1.3].

Lemma 2.52. — *Let $1 \leqslant p < \infty$ and $0 < s < 1$. Let $u \in \mathrm{W}^{s,p}(\mathbb{T};\mathbb{S}^1)$ and let $v \in \mathrm{W}^{s+1/p,p}(\mathbb{D};\mathbb{D})$ be its harmonic extension. Define*

$$d(\omega) \coloneqq \sup\left\{ r \in (0,1) \;\middle|\; |v(r\omega)| \leqslant \frac{1}{2} \right\}, \quad \forall\, \omega \in \mathbb{S}^1$$

$\Big($*with the convention $d(\omega) = 0$ if $|v(r\omega)| > \dfrac{1}{2}$ for every r*$\Big)$*. Then*

$$\int_{\mathbb{S}^1} \frac{1}{(1-d(\omega))^{sp}}\,\mathrm{d}\omega \lesssim |u|^p_{\mathrm{W}^{s,p}} + 1. \tag{2.202}$$

Proof of Lemma 2.52. — We may estimate the integral in (2.202) as follows:

$$\int_{\mathbb{S}^1} \frac{1}{(1-d(\omega))^{sp}}\,\mathrm{d}\omega \lesssim \int_{\{d(\omega)>1/2\}} \frac{1}{(1-d(\omega))^{sp}}\,\mathrm{d}\omega + 1.$$

Thus it suffices to consider the ω's such that $d(\omega) > \dfrac{1}{2}$ and to prove, instead of (2.202), that

$$\int_{\{d(\omega)>1/2\}} \frac{1}{(1-d(\omega))^{sp}}\,\mathrm{d}\omega \lesssim |u|^p_{\mathrm{W}^{s,p}}. \tag{2.203}$$

We next note the following norm equivalence. In the domain $\mathbb{D}\setminus\overline{\mathbb{D}}_{1/2}$ (where $\overline{\mathbb{D}}_{1/2}$ is the disc $\left\{x\in\mathbb{C}\;\middle|\;|x|\leqslant\dfrac{1}{2}\right\}$) we have

$$(2.204)\quad |v|^p_{\mathrm{W}^{s+1/p,p}(\mathbb{D}\setminus\mathbb{D}_{1/2})}\approx\int_{\mathbb{S}^1}|v(\cdot\omega)|^p_{\mathrm{W}^{s+1/p,p}((1/2,1))}\,\mathrm{d}\omega+\int_{1/2}^1|v(r\cdot)|^p_{\mathrm{W}^{s+1/p,p}(\mathbb{S}^1)}.$$

The above equivalence is standard in the flat case, where $\mathbb{D}\setminus\mathbb{D}_{1/2}$ is replaced by $\mathbb{R}^n\times\left(\dfrac{1}{2},1\right)$, and $\mathbb{S}^1$ is replaced by $\mathbb{R}^n\times\{1\}$ ([**1**, Theorem 7.46]). Estimate (2.204) is a straightforward variant of its "flat analog". We now note that (2.204) implies that for a.e. $\omega\in\mathbb{S}^1$, the map $\left(\dfrac{1}{2},1\right)\ni r\mapsto v(r\omega)$ belongs to $\mathrm{W}^{s+1/p,p}\left(\left(\dfrac{1}{2},1\right)\right)$, and the latter space embeds into $\mathrm{C}^{0,s}\left(\left[\dfrac{1}{2},1\right]\right)$. In particular, for a.e. $\omega\in\mathbb{S}^1$ we have $d(\omega)<1$. Therefore, we have

$$(2.205)\quad \begin{aligned}\frac{|v(\omega)-v(d(\omega)\omega)|}{(1-d(\omega))^s}&\leqslant\sup_{r,t\in[1/2,1]}\frac{|v(r\omega)-v(t\omega)|}{|r-t|^s}=|v(\cdot\omega)|_{\mathrm{C}^{0,s}([1/2,1])}\\&\lesssim|v(\cdot\omega)|_{\mathrm{W}^{s+1/p,p}((1/2,1))}.\end{aligned}$$

Since $|v(\omega)-v(d(\omega)\omega)|\geqslant|v(\omega)|-|v(d(\omega)\omega)|=\dfrac{1}{2}$, we obtain from (2.205) that

$$(2.206)\quad \frac{1}{(1-d(\omega))^{sp}}\lesssim\frac{1}{|v(\omega)-v(d(\omega)\omega)|^p}|v(\cdot\omega)|^p_{\mathrm{W}^{s+1/p,p}((1/2,1))}\lesssim|v(\cdot\omega)|^p_{\mathrm{W}^{s+1/p,p}((1/2,1))}.$$

Integrating the above estimate, and using (2.204), we find that

$$(2.207)\quad \int_{\mathbb{S}^1}\frac{1}{(1-d(\omega))^{sp}}\,\mathrm{d}\omega\lesssim\int_{\mathbb{S}^1}|v(\cdot\omega)|^p_{\mathrm{W}^{s+1/p,p}((1/2,1))}\,\mathrm{d}\omega\lesssim|v|^p_{\mathrm{W}^{s+1/p,p}(\mathbb{D}\setminus\mathbb{D}_R)}\leqslant|v|^p_{\mathrm{W}^{s+1/p,p}(\mathbb{D})}.$$

Finally, since the Poisson extension operator is bounded from $\mathrm{W}^{s,p}(\mathbb{S}^1)$ onto $\mathrm{W}^{s+1/p,p}(\mathbb{D})$ ([**53**, Thm. 4.3.3 (i)]), we have $|v|_{\mathrm{W}^{s+1/p,p}(\mathbb{D})}\lesssim|u|_{\mathrm{W}^{s,p}(\mathbb{S}^1)}$, which combined with (2.207) gives (2.203). □

Proof of Lemma 2.51. — We start by establishing (2.201) when $|u|_{\mathrm{W}^{s,p}}\ll1$. In this case, by the continuous embedding $\mathrm{W}^{s,p}(\mathbb{S}^1)\hookrightarrow\mathrm{VMO}(\mathbb{S}^1)$ (valid when $sp\geqslant1$), we also have $|u|_{\mathrm{BMO}}\ll1$. Next we use the following property of the harmonic extension v of an $\mathbb{S}^1$-valued function:

$$\operatorname{dist}\left(v(x),\mathbb{S}^1\right)\lesssim|u|_{\mathrm{BMO}},\quad\forall\,x\in\mathbb{D}$$

(see Lemma 2.54 below). By combining the above estimate with the fact that $|u|_{\mathrm{BMO}}\ll1$, we find that $\operatorname{dist}\left(v(x),\mathbb{S}^1\right)\leqslant\dfrac{1}{2}$, $\forall\,x\in\mathbb{D}$. Therefore, $|v(x)|\geqslant\dfrac{1}{2}$, $\forall\,x\in\mathbb{D}$. Recalling the definition of $\widetilde{u}$, this implies that $|\widetilde{u}|=1$ in $\mathbb{D}$ and thus $|\operatorname{Jac}\widetilde{u}(x)|=0$, $\forall x\in\mathbb{D}$. Thus the estimate (2.201) is trivially satisfied when $|u|_{\mathrm{W}^{s,p}}\ll1$.

Suppose now that $|u|_{\mathrm{W}^{s,p}}$ is greater than some constant C. In this case, it suffices to prove, instead of (2.201), the following weaker estimate

$$(2.208)\quad \int_{\mathbb{S}^1}\left(\int_0^1|\operatorname{Jac}\widetilde{u}(r\omega)|\,\mathrm{d}r\right)^{sp}\mathrm{d}\omega\lesssim|u|^p_{\mathrm{W}^{s,p}}+1.$$

Again considering the definition of $\widetilde{u}$, we note that $|\operatorname{Jac}\widetilde{u}(x)|\lesssim|\nabla v(x)|^2$, and that the Jacobian $\operatorname{Jac}\widetilde{u}(x)$ vanishes whenever $|v(x)|>\dfrac{1}{2}$. Since the map $v\colon\mathbb{D}\to\mathbb{D}$ is harmonic, its gradient satisfies the estimate $|\nabla v(x)| \leqslant \dfrac{1}{\operatorname{dist}(x, \mathbb{S}^1)} = \dfrac{1}{1-|x|}$. Consequently, using the notation $d(\omega)$ given in Lemma 2.52, we have

$$\begin{aligned}\int_0^1 |\operatorname{Jac}\widetilde{u}(r\omega)|\,\mathrm{d}r &= \int_0^{d(\omega)} |\operatorname{Jac}\widetilde{u}(r\omega)|\,\mathrm{d}r \lesssim \int_0^{d(\omega)} |\nabla v(r\omega)|^2\,\mathrm{d}r \\ &\leqslant \int_0^{d(\omega)} \frac{1}{(1-r)^2}\,\mathrm{d}r = \frac{1}{1-d(\omega)}.\end{aligned} \tag{2.209}$$

Using (2.209) together with Lemma 2.52, we obtain (2.208). □

Remark 2.53. — *By the Gagliardo-Nirenberg embedding* $\mathrm{W}^{s,p} \cap \mathrm{L}^\infty \hookrightarrow \mathrm{W}^{\theta s, p/\theta}$, $0 < \theta < 1$ *(valid except when* $s = p = 1$*), it is possible to remove the condition* $s < 1$ *in the statement of the Lemma 2.51.*

In contrast, if we remove the condition $sp \geqslant 1$*, then the first part of the proof of Lemma 2.51 does not hold anymore. However, the second part of the proof is still valid, and leads to the weaker conclusion* $\|\varepsilon\|_{\mathrm{L}^{sp}} \lesssim |u|_{\mathrm{W}^{s,p}}^{1/s} + 1$ *(valid whether the semi-norm* $|u|_{\mathrm{W}^{s,p}}$ *is small or not).*

The next result was used in the proof of Lemma 2.51.

Lemma 2.54. — *Let* $u \in \mathrm{VMO}(\mathbb{S}^1; \mathbb{S}^1)$ *and let* $v\colon \mathbb{D} \to \mathbb{D}$ *be its harmonic extension to* $\mathbb{D}$*. Then*

$$\operatorname{dist}\big(v(x), \mathbb{S}^1\big) \lesssim |u|_{\mathrm{BMO}}, \quad \forall\, x \in \mathbb{D}.$$

Proof. — Let $I(\omega,\delta) \coloneqq B_\delta(\omega) \cap \mathbb{S}^1$, $\forall\, \omega \in \mathbb{S}^1$, $\forall\, 0 < \delta < 1$. By [**20**, Lemma A3.1], there exists an $R \in (0,1)$ such that

$$\left| v(r\omega) - \fint_{I(\omega,1-r)} u(x)\,\mathrm{d}x \right| \lesssim |u|_{\mathrm{BMO}}, \quad \forall\, r > R, \forall\, \omega \in \mathbb{S}^1; \tag{2.210}$$

a crucial point is that R does not depend on u.

On the other hand, from [**19**, equation (7), p. 206] we have

$$\operatorname{dist}\left(\fint_{I(\omega,\delta)} u(x)\,\mathrm{d}x, \mathbb{S}^1 \right) \lesssim |u|_{\mathrm{BMO}}, \quad \forall\, \omega \in \mathbb{S}^1, \forall\, \delta \leqslant 2. \tag{2.211}$$

By combining (2.210) and (2.211) we find that

$$\operatorname{dist}\big(v(r\omega), \mathbb{S}^1\big) \lesssim |u|_{\mathrm{BMO}}, \quad \forall\, r > R, \forall\, \omega \in \mathbb{S}^1. \tag{2.212}$$

It remains to obtain the conclusion of the lemma when $|x| \leqslant R$. For this purpose, we proceed as follows. We integrate the inequality $\operatorname{dist}(v(x), \mathbb{S}^1) \leqslant |v(x) - u(z)|$, $\forall\, z \in \mathbb{S}^1$, and find that

$$\operatorname{dist}\big(v(x), \mathbb{S}^1\big) \leqslant \fint_{\mathbb{S}^1} |v(x) - u(z)|\,\mathrm{d}z = \fint_{\mathbb{S}^1} \left| \int_{\mathbb{S}^1} P(x,y) u(y)\,\mathrm{d}y - u(z) \right| \mathrm{d}z;$$

we recall that $P(x,y)$ denotes the Poisson kernel. Since $\displaystyle\int_{\mathbb{S}^1} P(x,y)\,\mathrm{d}y \equiv 1$ and $|x| \leqslant R$, we find that

$$\begin{aligned}\operatorname{dist}\big(v(x),\mathbb{S}^1\big) &\leqslant \fint \left|\int_{\mathbb{S}^1} P(x,y)[u(y)-u(z)]\,\mathrm{d}y\right| \mathrm{d}z\\ &\lesssim \fint_{\mathbb{S}^1}\fint_{\mathbb{S}^1} P(x,y)|u(y)-u(z)|\,\mathrm{d}y\mathrm{d}z\\ &\lesssim \fint_{\mathbb{S}^1}\fint_{\mathbb{S}^1} |u(y)-u(z)|\,\mathrm{d}y\mathrm{d}z \lesssim |u|_{\mathrm{BMO}},\end{aligned}$$

where the next to the last inequality comes from the fact that $P(x,y)$ is uniformly bounded when $|x| \leqslant R$ and $y \in \mathbb{S}^1$. This estimate together with (2.212) concludes the proof. □

We now establish the following result, in the spirit of the theory of weighted Sobolev spaces ([**54**], [**53**, Section 2.12.2, Theorem, p. 184]). For related results, see also [**40**], [**45**].

Lemma 2.55. — *Let $1 \leqslant p < \infty$ and $0 < s < 1$. Given $u \in C^\infty(\mathbb{T};\mathbb{C})$, let v be the harmonic extension of u and let $\widetilde{u}$ be given by* (2.89)*. Let $\delta(x) := 1-|x|$, $\forall\, x \in \mathbb{D}$. Then*

$$\int_{\mathbb{D}} \delta(x)^{p-sp-1}|\nabla v(x)|^p\,\mathrm{d}x \lesssim |u|^p_{W^{s,p}} \text{ and } \int_{\mathbb{D}} \delta(x)^{p-sp-1}|\nabla \widetilde{u}(x)|^p\,\mathrm{d}x \lesssim |u|^p_{W^{s,p}}. \tag{2.213}$$

Proof. — We start by noting that it suffices to prove the first inequality in (2.213). Indeed, we have $\widetilde{u} = \Pi \circ v$, with Π smooth, and therefore $|\nabla\widetilde{u}| \lesssim |\nabla v|$. Therefore, the second estimate in (2.213) is a consequence of the first one.

Let $P(x,y)$ denote the Poisson kernel in the unit disc $\mathbb{D}$. Since v is the harmonic extension of u in $\mathbb{D}$ and $\displaystyle\int_{\mathbb{S}^1} \nabla_x P(x,y)\,\mathrm{d}y = 0$ (following by differentiating the identity $\displaystyle\int_{\mathbb{S}^1} P(x,y)\,\mathrm{d}y \equiv 1$), we have

$$\nabla v(x) = \int_{\mathbb{S}^1} \nabla_x P(x,y)u(y)\,\mathrm{d}y = \int_{\mathbb{S}^1} \nabla_x P(x,y)[u(y)-u(\omega)]\,\mathrm{d}y, \quad \forall\,\omega \in \mathbb{S}^1. \tag{2.214}$$

We next pass to polar coordinates into the first integral in (2.213). Using the fact that $r \leqslant 1$, we obtain

$$\int_{\mathbb{D}} \delta(x)^{p-sp-1}|\nabla v(x)|^p\,\mathrm{d}x \leqslant \int_0^1\int_{\mathbb{S}^1} \delta(r\omega)^{p-sp-1}|\nabla v(r\omega)|^p\,\mathrm{d}\omega\mathrm{d}r. \tag{2.215}$$

We next estimate $|\nabla v(r\omega)|$. For this purpose, we rely on the following properties of $\nabla_x P(x,y)$:

$$|\nabla_x P(x,y)| \lesssim \frac{1}{\delta(x)^2}, \quad \forall\, x \in \mathbb{D}, \forall\, y \in \mathbb{S}^1 \tag{2.216}$$

and

$$|\nabla_x P(r\omega,y)| \lesssim \frac{1}{|y-\omega|^2},\ \forall\,\omega \in \mathbb{S}^1, \forall\, r \in [0,1), \forall\, y \in \mathbb{S}^1 \text{ such that } |y-\omega| \geqslant \delta(r\omega). \tag{2.217}$$

The above inequalities are obtained as follows. We start from the straightforward estimates

$$|\nabla_x P(x,y)| \lesssim \frac{1}{|x-y|^2} + \frac{1-|x|}{|x-y|^3} \lesssim \frac{1}{|x-y|^2}. \tag{2.218}$$

Then (2.216) is a consequence of (2.218) combined with $|x-y| \geqslant \delta(x)$. On the other hand, we have $|y-r\omega| \geqslant 1-r$ and therefore

$$|y-\omega| \leqslant |y-r\omega| + |\omega - r\omega| = |y-r\omega| + 1 - r \leqslant 2|y-r\omega|. \tag{2.219}$$

Estimate (2.217) is a consequence of (2.218) and of (2.219).

We return to (2.214) and we split the integral as follows:

$$\begin{aligned} |\nabla v(r\omega)|^p &\leqslant \left(\int_{\mathbb{S}^1} |\nabla_x P(r\omega, y)| |u(y) - u(\omega)| \, \mathrm{d}y \right)^p \\ &\lesssim \left(\int_{|y-\omega| \leqslant \delta(r\omega)} |\nabla_x P(r\omega, y)| |u(y) - u(\omega)| \, \mathrm{d}y \right)^p \\ &+ \left(\int_{|y-\omega| \geqslant \delta(r\omega)} |\nabla_x P(r\omega, y)| |u(y) - u(\omega)| \, \mathrm{d}y \right)^p \coloneqq I_1(r,\omega) + I_2(r,\omega). \end{aligned} \tag{2.220}$$

Then estimates (2.215) and (2.220) lead to

$$\int_{\mathbb{D}} \delta(x)^{p-sp-1} |\nabla v(x)|^p \, \mathrm{d}x \lesssim \int_0^1 \int_{\mathbb{S}^1} \delta(r\omega)^{p-sp-1} [I_1(r,\omega) + I_2(r,\omega)] \, \mathrm{d}\omega \mathrm{d}r \coloneqq J_1 + J_2. \tag{2.221}$$

It remains to estimate J_1 and J_2.

Using (2.216) and Hölder's inequality we find

$$\begin{aligned} I_1(r,\omega) &\lesssim \left(\int_{|y-\omega| \leqslant \delta(r\omega)} \frac{|u(y) - u(\omega)|}{\delta(r\omega)^2} \, \mathrm{d}y \right)^p = \frac{1}{\delta(r\omega)^{2p}} \left(\int_{|y-\omega| \leqslant \delta(r\omega)} |u(y) - u(\omega)| \, \mathrm{d}y \right)^p \\ &\lesssim \frac{1}{\delta(r\omega)^{2p}} \delta(r\omega)^{p-1} \int_{|y-\omega| \leqslant \delta(r\omega)} |u(y) - u(\omega)|^p \, \mathrm{d}y \\ &= \frac{1}{\delta(r\omega)^{p+1}} \int_{|y-\omega| \leqslant \delta(r\omega)} |u(y) - u(\omega)|^p \, \mathrm{d}y. \end{aligned}$$

Inserting the above estimate of $I_1(r,\omega)$ in the expression of J_1, we find that

$$\begin{aligned} J_1 &\lesssim \int_0^1 \int_{\mathbb{S}^1} \frac{1}{\delta(r\omega)^{sp+2}} \int_{|y-\omega| \leqslant \delta(r\omega)} |u(y) - u(\omega)|^p \, \mathrm{d}y \mathrm{d}\omega \mathrm{d}r \\ &= \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \left(\int_0^{1-|y-\omega|} \frac{1}{(1-r)^{sp+2}} \, dr \right) |u(y) - u(\omega)|^p \, \mathrm{d}y \mathrm{d}\omega \\ &\lesssim \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} \frac{|u(y) - u(\omega)|^p}{|y-\omega|^{sp+1}} \, \mathrm{d}y \mathrm{d}\omega = |u|^p_{\mathrm{W}^{s,p}}. \end{aligned} \tag{2.222}$$

Similarly, for I_2 we use (2.217) and Hölder's inequality as follows:

$$\begin{aligned} I_2(r,\omega) &\lesssim \left(\int_{|y-\omega| > \delta(r\omega)} \frac{1}{|y-\omega|^2} |u(y) - u(\omega)| \, \mathrm{d}y \right)^p \\ &= \left(\int_{|y-\omega| > \delta(r\omega)} \frac{|u(y) - u(\omega)|}{|y-\omega|^{2-\alpha}} |y-\omega|^{-\alpha} \, \mathrm{d}y \right)^p \\ &\leqslant \int_{|y-\omega| > \delta(r\omega)} \frac{|u(y) - u(\omega)|^p}{|y-\omega|^{(2-\alpha)p}} \, \mathrm{d}y \left(\int_{|y-\omega| > \delta(r\omega)} |y-\omega|^{-\alpha p/(p-1)} \, \mathrm{d}y \right)^{p-1}. \end{aligned} \tag{2.223}$$

Assuming that $\alpha > 1 - \dfrac{1}{p}$, the last integral in (2.223) can be estimated as follows:

$$\int_{|y-\omega|>\delta(r\omega)} |y-\omega|^{-\alpha p/(p-1)}\,\mathrm{d}y \lesssim \delta(r\omega)^{-\alpha p/(p-1)+1}.$$

Hence, returning to (2.223), we have

$$I_2(r,\omega) \lesssim \delta(r\omega)^{-\alpha p+p-1}\int_{|y-\omega|>\delta(r\omega)} \frac{|u(y)-u(\omega)|^p}{|y-\omega|^{(2-\alpha)p}}\,\mathrm{d}y.$$

Using the above estimate of $I_2(r,\omega)$ in J_2 we find

$$\begin{aligned}J_2 &\lesssim \int_0^1\int_{\mathbb{S}^1} \delta(r\omega)^{2p-sp-2-\alpha p}\int_{|y-\omega|\geqslant\delta(r\omega)} \frac{|u(y)-u(\omega)|^p}{|y-\omega|^{(2-\alpha)p}}\,\mathrm{d}y\mathrm{d}\omega\mathrm{d}r\\ &= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\left(\int_{1-|y-\omega|}^1 (1-r)^{2p-sp-2-\alpha p}\,\mathrm{d}r\right)\frac{|u(y)-u(\omega)|^p}{|y-\omega|^{(2-\alpha)p}}\,\mathrm{d}y\mathrm{d}\omega.\end{aligned} \tag{2.224}$$

By the above, if we choose $\alpha \in \left(1-\dfrac{1}{p}, 2-s-\dfrac{1}{p}\right)$, then we obtain $J_2 \lesssim |u|^p_{\mathrm{W}^{s,p}}$. This estimate, together with (2.222) and (2.221), leads to $\displaystyle\int_{\mathbb{D}} \delta(x)^{p-sp-1}|\nabla v(x)|^p\,\mathrm{d}x \lesssim |u|^p_{\mathrm{W}^{s,p}}$. □

2.7.7.6. Toolbox for "further thoughts". — This subsection contains the lemmas needed in Subsection 2.7.5.

We start by proving that property (R) discussed in Subsection 2.7.5.1 holds in the following weaker form.

Lemma 2.56. — *Let $0 < s < \infty$ and $1 \leqslant p < \infty$ be such that s and sp are not integers. Then*

$$\mathrm{W}^{s,p}((0,1)^n) = \left(\mathrm{W}^{s,p}((0,1)^n)\cap B^0_{\infty,\infty}((0,1)^n)\right) + \left(\mathrm{W}^{s,p}((0,1)^n)\cap \mathrm{W}^{sp,1}((0,1)^n)\right). \tag{2.225}$$

Proof. — By a standard extension argument, it suffices to prove that the above holds when $(0,1)^n$ is replaced by $\mathbb{R}^n$.

In order to obtain the analog of (2.225) in the whole $\mathbb{R}^n$, we rely on Littlewood-Paley decompositions (but alternatively, we could use wavelets as in [**36**]). Let η and λ be two maps in $\mathrm{C}^\infty_c(B(0,1);\mathbb{R}_+)$ such that $\eta = 1$ in $\overline{B}\left(0,\dfrac{4}{5}\right)$ and $\lambda \equiv 1$ in $\overline{B}\left(0,\dfrac{4}{5}\right)\setminus B\left(0,\dfrac{3}{8}\right)$. Define ψ_j, $j \geqslant 0$, by

$$\widehat{\psi_0}(\xi) \coloneqq \eta(\xi) \quad\text{and, for every} j\geqslant 1,\quad \widehat{\psi_j}(\xi) \coloneqq \lambda\left(\xi/2^j\right).$$

It is easy to check that, with φ_j given by (2.186) and (2.187), we have $\widehat{\varphi_j}\widehat{\psi_j} = \widehat{\varphi_j}$, and thus

$$\varphi_j * \psi_j = \varphi_j, \quad \forall\, j. \tag{2.226}$$

On the other hand, we have

$$\|\psi_j\|_{\mathrm{L}^1} = \|(\mathcal{F}^{-1}\lambda)_{2^{-j}}\|_{\mathrm{L}^1} = \|\mathcal{F}^{-1}\lambda\|_{\mathrm{L}^1}, \quad \forall\, j \geqslant 1. \tag{2.227}$$

Let $f \in \mathrm{W}^{s,p}$. We split $f_j = g_j + h_j$, where $g_j \coloneqq f_j\mathbb{1}_{\{|f_j|\leqslant 1\}}$ and $h_j \coloneqq f_j\mathbb{1}_{\{|f_j|>1\}}$. Clearly,

$$\|g_j\|_{\mathrm{L}^p} \leqslant \|f_j\|_{\mathrm{L}^p}, \quad \|g_j\|_{\mathrm{L}^\infty} \leqslant 1, \quad \|h_j\|_{\mathrm{L}^1} \leqslant \|f_j\|^p_{\mathrm{L}^p}. \tag{2.228}$$

Using (2.226), (2.227) and (2.228), we obtain

$$f_j = f_j * \psi_j = g_j * \psi_j + h_j * \psi_j := G_j + H_j,$$

with

$$\|G_j\|_{\mathrm{L}^p} \lesssim \|f_j\|_{\mathrm{L}^p}, \quad \|G_j\|_{\mathrm{L}^\infty} \lesssim 1, \quad \|H_j\|_{\mathrm{L}^1} \lesssim \|f_j\|^p_{\mathrm{L}^p} \tag{2.229}$$

and

$$\operatorname{supp}\widehat{G_j}, \quad \operatorname{supp}\widehat{H_j} \subset \operatorname{supp}\widehat{\psi_j} \subset \mathcal{C}_j \cup \mathcal{C}_{j-1}. \tag{2.230}$$

By (2.191), (2.192), (2.229) and (2.230), we find that $f = g + h$, where $g := \sum G_j$ and $h := \sum H_j$ satisfy

$$g \in \mathrm{W}^{s,p} \cap B^0_{\infty,\infty}, \; h \in \mathrm{W}^{s,p} \cap \mathrm{W}^{sp,1}, \; \|g\|_{B^0_{\infty,\infty}} \lesssim 1, \; \|g\|^p_{\mathrm{W}^{s,p}} + \; \|h\|_{\mathrm{W}^{sp,1}} \lesssim \|f\|^p_{\mathrm{W}^{s,p}}. \quad \square \tag{2.231}$$

We now prove Lemma 2.19, used in Subsection 2.7.5.2 for constructing a lifting in $\mathrm{W}^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ when $sp < 1$.

Proof of Lemma 2.19. — Assume first that U is smooth in $\mathbb{T}^n \times [0,1]$. In this case we have $f(x) = U(x,0)$. Let $x, y \in \mathbb{T}^n$ and set $r := |y-x| \in [0,1]$ and $\omega := \dfrac{y-x}{|y-x|}$, which satisfies $|\omega| = 1$. We have

$$\begin{aligned}|f(y)-f(x)| &\leqslant |f(y)-U(y,r)| + |f(x)-U(x,r)| + |U(y,r)-U(x,r)|\\ &\leqslant \int_0^r |\nabla U(y,\varepsilon)|\,\mathrm{d}\varepsilon + \int_0^r |\nabla U(x,\varepsilon)|\,\mathrm{d}\varepsilon + \int_0^r |\nabla U(x+\varepsilon\omega, r)|\,\mathrm{d}\varepsilon\\ &=: F(x,r,\omega).\end{aligned} \tag{2.232}$$

Integrating (2.232), we find that

$$\int_{\mathbb{T}^n} |f(x+r\omega) - f(x)|\,\mathrm{d}x \leqslant \int_{\mathbb{T}^n} F(x,r,\omega)\,\mathrm{d}x. \tag{2.233}$$

Assume next that U is not necessarily smooth up to $\varepsilon = 0$. Then we may assume that

$$\int_{\mathbb{T}^n\times(0,1)} \varepsilon^{-\sigma} |\nabla U(x,\varepsilon)|\,\mathrm{d}x\mathrm{d}\varepsilon < 0,$$

for otherwise there is nothing to prove. Then $U \in \mathrm{W}^{1,1}(\mathbb{T}^n \times (0,1))$. By a standard approximation procedure, we find that (with $f = \operatorname{tr} U$) inequality (2.233) still holds for such U.

By combining (2.233) with the formula of the $\mathrm{W}^{\sigma,1}$ semi-norm and passing to spherical coordinates, we find that

$$
\begin{aligned}
|f|_{\mathrm{W}^{\sigma,1}(\mathbb{T}^n)} &= \int_{\mathbb{T}^n\times\mathbb{T}^n} \frac{|f(y)-f(x)|}{|y-x|^{n+\sigma}}\,\mathrm{d}x\mathrm{d}y \\
&\lesssim \int_{\mathbb{T}^n\times\mathbb{T}^n} \frac{1}{|y-x|^{n+\sigma}} \left(\int_0^{|y-x|} |\nabla U(x,\varepsilon)|\,\mathrm{d}\varepsilon\right) \mathrm{d}x\mathrm{d}y \\
&+ \int_{\mathbb{T}^n\times\mathbb{T}^n} \frac{1}{|y-x|^{n+\sigma}} \left(\int_0^{|y-x|} \left|\nabla U\left(x+\frac{\varepsilon(y-x)}{|y-x|},|y-x|\right)\right| \mathrm{d}\varepsilon\right) \mathrm{d}x\mathrm{d}y \\
&\lesssim \int_{\mathbb{T}^n}\int_0^1 \frac{1}{r^{1+\sigma}} \int_0^r |\nabla U(x,\varepsilon)|\,\mathrm{d}\varepsilon\mathrm{d}r\mathrm{d}x \\
&+ \int_{\mathbb{T}^n}\int_0^1 \int_{\mathbb{S}^{n-1}} \frac{1}{r^{1+\sigma}} \int_0^r |\nabla U(x+\varepsilon\omega,r)|\,\mathrm{d}\varepsilon\mathrm{d}\omega\mathrm{d}r\mathrm{d}x \\
&\lesssim \int_{\mathbb{T}^n\times(0,1)} \left(\int_\varepsilon^1 \frac{1}{r^{1+\sigma}}\,dr\right) |\nabla U(x,\varepsilon)|\,\mathrm{d}x\mathrm{d}\varepsilon \\
&+ \int_{(0,1)} \frac{1}{r^{1+\sigma}} \left(\int_{\mathbb{S}^{n-1}\times(0,r)} \left(\int_{\mathbb{T}^n} |\nabla U(x+\varepsilon\omega,r)|\,\mathrm{d}x\right) \mathrm{d}\omega\mathrm{d}\varepsilon\right) \mathrm{d}r \\
&\lesssim \int_{\mathbb{T}^n\times(0,1)} \varepsilon^{-\sigma}|\nabla U(x,\varepsilon)|\,\mathrm{d}x\mathrm{d}\varepsilon,
\end{aligned}
$$

that is, (2.100) holds and Lemma 2.19 is proven. □

ABSTRACT

Dans cette thèse nous étudions quelques aspects des certains espaces de fonctions. Plus précisément, nous nous intéressons aux singularités des applications $\mathrm{W}^{1,n}$ à valeurs dans la sphère $\mathbb{S}^n$ et aux relèvements des applications $\mathrm{W}^{s,p}$ à valeurs dans le cercle $\mathbb{S}^1$. La thèse contient deux parties, associées chacune à l'une des thèmes précédentes.

La première partie concerne le problème de minimisation suivant :

$$\inf \int_\Omega a(x)|\mathrm{D}u(x)|^n \,\mathrm{d}x,$$

où les fonctions admissibles sont : $u \in \mathrm{W}^{1,n}(\Omega;\mathbb{S}^n)$, pour un domaine $\Omega \subset \mathbb{R}^p$ borné et lisse, qui vérifient $\star\mathrm{J}u = \Gamma\, \mathcal{H}^n(\mathbb{S}^n)/(n+1)$. La fonction $a(\cdot)$, qui est continue et positive, représente un poids fixé. Le Jacobien $\mathrm{J}u$ fourni l'ensemble des singularités topologiques de u, qui empêchent que u soit approchable par des fonctions lisses, et il est prescrit en termes d'un courant rectifiable Γ de dimension $m \coloneqq p-n$ donnée.

Nous obtenons une formule exacte de cet infimum :

$$n^{n/2}\mathcal{H}^n(\mathbb{S}^n)\ \inf\{\mathrm{mass}(M \llcorner a)\ ;\ M \text{ courant rectifiable de bord } \Gamma\}.$$

Ce résultat améliore un résultat d'Alberto, Baldi et Orlandi de 2003. A son tour, ce résultat est une extension dans une liste des généralisations considérées à partir d'un problème proposé et étudié par Brezis, Coron et Lieb en 1986, et par Almgren, Browder et Lieb en 1988.

La deuxième partie porte sur le problème suivant : trouver la meilleure estimation de la forme

$$|\varphi|_{\mathrm{W}^{s,p}} \leqslant F(|u|_{\mathrm{W}^{s,p}}),$$

où φ est un relèvement d'une application u à valeurs dans $\mathbb{C}$ donnée, c.-à-d. $u \equiv \exp(\imath\varphi)$. La fonction F va dépendre de s et de p, et ce qu'on entend par “meilleure” estimation dépend du type des estimations correspondants aux s et p donnés.

Dans le cas $sp < 1$ nous obtenons l'estimation optimale sous la forme

$$|\varphi|_{\mathrm{W}^{s,p}} \leqslant C\frac{1}{s(1-sp)}|u|_{\mathrm{W}^{s,p}},$$

ce qui représente une généralisation des résultats obtenus par Bourgain, Brezis et Mironescu en 2000 et 2002 pour le cas $p=2$. A l'aide des mêmes méthodes utilisées pour la preuve du résultat mentionné, on obtient en plus une semi-norme dyadique sur les espaces de Sobolev fractionnaires $\mathrm{W}^{s,p}$ avec $0 < s < 1$, équivalente à la semi-norme habituelle. Pour le cas $sp \geqslant 1$ nous obtenons les estimations optimales suivantes :

$$|\varphi|_{\mathrm{W}^{s,p}} \leqslant C|u|_{\mathrm{W}^{s,p}} \text{ si } s \geqslant 1 \quad \text{et} \quad |\varphi|_{\mathrm{W}^{s,p}} \leqslant C\left(|u|_{\mathrm{W}^{s,p}} + |u|^{1/s}_{\mathrm{W}^{s,p}}\right) \text{ si } s < 1.$$

Cela représente une extension d'un résultat de Merlet de 2006 aux dimensions supérieures.

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